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Research Article

Graphical Exploration of Generalized Picture Fuzzy Hypersoft Information with Application in Human Resource Management Multiattribute Decision-Making

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In this study, the graphical exploration of a novel hybrid, that is, possibility of picture fuzzy hypersoft graph (popfhs-graph) is accomplished. The popfhs-graph is more flexible and reliable in the sense that it has the ability to tackle the limitations of picture fuzzy soft graph regarding the entitlement of multiargument approximate mapping and possibility degree-based setting. Its approximate mapping considers the Cartesian product of subclasses of parameters as domain and then maps it to the power set of universal set. The possibility degree-based setting ensures the assessment of uncertain attitude of approximate elements up to the level of acceptance. First, some of elementary notions and set theoretic operations of popfhs-graph are investigated with the support of numerical examples and pictorial representations. Second, some of its variants, types of products, and composition are also discussed. Lastly, based on aggregation of the popfhs-graph, an algorithm is proposed for multiattribute decision-making problem and validated by resolving daily-life recruitment problem for the best selection of candidate.

1. Introduction

The information that are obtained for any kind of computation have some kinds of vagueness and uncertainty. In this regard, fuzzy set (f-set) [1] and intuitionistic fuzzy set (if-set) [2] are initiated to tackle such type of informational uncertainty and vagueness. In f-set, the condition "well defined" of classical set is characterized by a membership function μ_T defined by a membership grade $\mu_T(u_i)$ within $[0, 1]$ for all members u_i of initial universe $\mathscr{U}$, whereas the if-set characterizes such condition by two functions, that is, membership function μ_T and nonmembership function μ_F defined by membership grade $\mu_T(u_i)$ and nonmembership grade $\mu_F(u_i)$, respectively, within $[0, 1]$ for all $u_i \in \mathscr{U}$ subject to conditions that both $\mu_T(u_i)$ and $\mu_F(u_i)$ are dependent and their sum $\mu_T(u_i) + \mu_F(u_i)$ must lie within $[0, 1]$ with hesitancy grade $\mu_H(u_i) = 1 - (\mu_T(u_i) + \mu_F(u_i))$. Both f-set and if-set are not capable to tackle the situations in which a neutral grade along with refusal grade is required to be emphasized, so picture fuzzy set (pf-set) [3] is conceptualized to manage such scenarios. In pf-set, objective belongingness is further characterized by three membership functions, that is, μ_T, μ_I, and μ_F such that for all $u_i \in \mathscr{U}$, their sum $\mu_T(u_i) + \mu_I(u_i) + \mu_F(u_i)$ and refusal grade $\mu_R(u_i) = 1 - \mu_T(u_i) - \mu_I(u_i) - \mu_F(u_i)$ lie within $[0, 1]$. Recently, the uncertain complexities in different daily-life problems have been investigated and addressed by the authors [4–8] through employing various multicriteria decision-making techniques based on pf-set. In order to equip f-set, if-set and pf-set with parameterization tool, Molodtsov [9] introduced soft set (s-set) for dealing with uncertainties and vagueness. The fuzzy soft set fs-set [10], intuitionistic fuzzy

soft set (ifs-set) [11], and picture fuzzy soft set (pfs-set) [12] are the hybridized structures of f-set, if-set, and pf-set, respectively, with s-set to tackle uncertainties with the support of parameterization tool. The researchers [13–15] discussed the applicability of pfs-set in various fields of study for solving many real-life decision-making based problems. Kamacı et al. [16] discussed multiperiod decision-making problem based on dynamic aggregation operators and Einstein operations of interval-valued pfs-set. Kamacı [17] applied the extended concept of pfs-set with linguistic information in game theory. Akcetin and Kamacı [18] used TOPSIS and ELECTRE decision-making techniques by using three-valued s-set.

In various real-world scenarios, the classification of attributes into subattributive values in the form of non-overlapping sets is necessary. The s-sets and its hybrids are incompatible with such scenarios, so Smarandache [19] introduced hypersoft sets (*hs*-sets) to address the insufficiency of s-set-like models. In *hs*-set, a new approximate function, multiargument approximate function (maa-function), is employed that maps Cartesian product of attribute-valued disjoint sets to the power set of initial universe. The rudiments and elementary axioms of *hs*-sets have been discussed in [20] and elaborated with numerical examples. Ihsan et al. [21] discussed the validity of *hs*-sets for the entitlement of multidecisive opinions under expert set environment. Rahman et al. [22–24] investigated the hybridized properties of *hs*-sets under the environments of convexity and concavity, parameterization, and bijection. They employed decision-making algorithmic approaches to solve real-world problems. The hybridized structures of *hs*-set with f-set and if-set are fuzzy hypersoft set (fhs-set) [25, 26] and intuitionistic fuzzy hypersoft set ($ifhs$-set) [27], respectively. Saeed et al. [28] characterized the concept of hypersoft graphs and discussed its some properties.

In order to manage s-set information in graphs, Thumbakara and George [29] characterized soft graphs in 2014. As its extension, fuzzy soft graph (fs-graph) [30], intuitionistic fuzzy soft graph (ifs-graph) [31], and picture fuzzy soft graph (pfs-graph) [32] are conceptualized. In order to evaluate uncertain nature of approximate elements as a whole in fs-set and ifs-set, Alkhazaleh et al. [33] and Bashir et al. [34] characterized possibility fuzzy soft set ($pofs$-set) and possibility intuitionistic fuzzy soft set ($poifs$-set), respectively, by assigning possibility degree to each approximate element collectively in these structures. Akram and Shahzadi [35] developed possibility intuitionistic fuzzy soft graph ($poifs$-graph) and investigated some of its properties and operations.

In $poifs$-graph, neutral membership grade is ignored, and it lacks the consideration of maa-function. Consequently, various daily-life scenarios such as recruitment process, medical diagnosis, optimal product selection, and so on are not tackled by $poifs$-graph that demands a novel graphical structure to be characterized in literature. We, therefore, present a new graphical model, that is, possibility picture fuzzy hypersoft graph (popfhs-graph) in this study. Its advantageous aspects are as follows: it provides due status to neutral membership degree while dealing uncertainties, and it employs maa-function to cope with the scenarios having further partitioning of attributes into their respective attribute-valued sets. The proposed graphical model is more flexible as it overcomes the shortcomings of existing relevant models for dealing with uncertainties. It assigns a possibility degree to each approximate element of its maa-function to deal with its uncertain behavior. The major contributions of the paper are outlined as follows:

(1) A novel graphical hybrid popfhs-graph is characterized that is capable to cope with the following situations collectively:
   (a) The situation in which the categorization of opted parameters into their respective disjoint subclasses having their relevant attributive values is necessary
   (b) The situation in which the consideration of multiargument parameterization in the domain of approximate mapping is mandatory to have reliable approximation of alternatives
   (c) The situation that demands a mode for the assessment of uncertain nature of approximate elements to assess the level of acceptance

(2) The novel notions of $pfhs$-set, popfhs-set, and popfhs-graph are introduced, and then some essential fundamental properties such as subgraph, spanning subgraph, strong subgraph; aggregation operations such as union, intersection, and complement; and products such as Cartesian product, cross product, lexicographic product, strong product, and composition of popfhs-graph are investigated.

(3) A decision-support system is constructed based on the proposal of an algorithm that is implemented in real-life multiattribute decision-making problem for the selection of suitable candidate.

The layout of the remaining paper is as follows: some essential definitions relevant to main work are recalled from existing literature for proper understanding of proposed study in Section 2. Some fundamentals, that is, properties and set-theoretic operations of popfhs-graph, are characterized with graphical representation-based examples in Section 3. Section 4 presents the analytical cum graphical exploration of some products and compositions of popfhs-graphs. In Section 5, the concept of popfhs-graph is applied in decision-making for HRM-scenario. The comparison analysis is presented in Section 6. Lastly, paper is summarized with more future directions in Section 7.

2. Preliminaries

This portion of the paper presents some elementary terms and definitions by reviewing the existing literature for vivid understanding of the proposed study.

In literature, Cuong [3] characterized the following concept of pf-set as a generalization of if-set [2] by introducing a new grade called neutral grade $\mu_N(\hat{u})$ to furnish the neutrality of decision-makers.

Definition 1 (see [3]). A pf-set $P_{\mathscr{F}}$ is defined as $P_{\mathscr{F}} = \{(\widehat{u}, < \mu_T(\widehat{u}), \mu_N(\widehat{u}), \mu_F >)|\widehat{u} \in u|\}$ such that $\mu_T(\widehat{u})$, $\mu_N(\widehat{u})$, $\mu_F(\widehat{u})$: $\mathscr{U} \longrightarrow [0,1]$, where $\mu_T(\widehat{u})$, $\mu_N(\widehat{u})$, and $\mu_F(\widehat{u})$ represent positive, neutral, and negative membership grades, respectively, of $\widehat{u} \in \mathscr{U}$ subject to the condition that $0 \leq \mu_T(\widehat{u}) + \mu_N(\widehat{u}) + \mu_F(\widehat{u}) \leq 1$ with refusal membership grade $\mu_R(\widehat{u}) = 1 - \mu_T(\widehat{u}) - \mu_N(\widehat{u}) - \mu_F(\widehat{u})$. The collection of all pf-sets over $\mathscr{U}$ is denoted as $\mathbb{P}_{\mathscr{F}}(\mathscr{U})$.

The parameters play a key role in reliable and authentic decision-making process. The existing fuzzy set-like models are inadequate with any kind of parameterization tool; therefore, Molodtsov [9] initiated the following idea of s-set to address the limitations of predefined uncertain models with the provision of parameterization tool.

Definition 2 (see [9]). A s-set $\mathbb{S}$ over $\mathscr{U}$ is a pair $(\psi_{\mathbb{S}}, \mathfrak{A})$, where $\psi_{\mathbb{S}}$: $\mathfrak{A} \longrightarrow \mathbb{P}(\mathscr{U})$ is an approximate function of $\mathbb{S}$ and $\mathfrak{A} \subseteq \mathfrak{E}$ (a set of parameters). For any $\widehat{a} \in \mathfrak{A}$, $\psi_{\mathbb{S}}(\widehat{a})$ is called an approximate element of $\mathbb{S}$. In this definition, the symbol $\mathbb{P}(\mathscr{U})$ is meant for power set of $\mathscr{U}$.

In 2015, Yang et al. [12] developed the following novel model, that is, pfs-set to tackle the insufficiencies of pf-set for parameterization context. In short, they combined the theory of pf-set with Molodtsov's theory of s-set to carve out a parameterized family of universal set with picture fuzzy setting.

Definition 3 (see [12]). A pfs-set $\mathbb{P}$ over $\mathscr{U}$ is a pair $(\psi_{\mathbb{P}}, \mathbb{Z})$, where $\psi_{\mathbb{P}}$: $\mathbb{Z} \longrightarrow \mathbb{P}_{\mathscr{F}}(\mathscr{U})$ and $\mathbb{Z} \subseteq \mathfrak{E}$. For any $\widehat{z} \in \mathbb{Z}$, $\psi_{\mathbb{P}}(\widehat{z})$ is a pfs-subset and known as approximate element of pfs-set $\mathbb{P}$, that is, $\psi_{\mathbb{P}}(\widehat{z})$ can be represented as pfs-set over $\mathscr{U}$ such that $\psi_{\mathbb{P}}(\widehat{z}) = \left\{(\widehat{u}, < \mu_T^{\widehat{z}}(\widehat{u}), \mu_N^{\widehat{z}}(\widehat{u}), \mu_F^{\widehat{z}}(\widehat{u}) >)|\widehat{u} \in \mathscr{U}\right\}$, where $\mu_T^{\widehat{z}}(\widehat{u})$, $\mu_N^{\widehat{z}}(\widehat{u})$, and $\mu_F^{\widehat{z}}(\widehat{u})$ represent positive, neutral, and negative membership grades, respectively, of $\widehat{u} \in \mathscr{U}$ subject to the condition that $0 \leq \mu_T^{\widehat{z}}(\widehat{u}) + \mu_N^{\widehat{z}}(\widehat{u}) + \mu_F^{\widehat{z}}(\widehat{u}) \leq 1$ with refusal membership grade $\mu_R^{\widehat{z}}(\widehat{u}) = 1 - \mu_T^{\widehat{z}}(\widehat{u}) - \mu_N^{\widehat{z}}(\widehat{u}) - \mu_F^{\widehat{z}}(\widehat{u})$.

In 2018, Smarandache [19] extended the concept of s-set and developed a novel model, that is, hs-set that utilizes a novel mapping known as maa-function to deal with the shortcomings of s-set regarding the categorization of parameters into their relevant parametric valued subclasses.

Definition 4 (see [19]). A hs-set $\mathfrak{H}$ over $\mathscr{U}$ is a pair $(\mathscr{W}, \mathscr{D})$, where $\mathscr{D}$ is the Cartesian product of $\mathscr{D}^i, i = 1, 2, 3, \ldots, n, \mathscr{D}^i \cap \mathscr{D}^j = \varnothing \, \forall \, i \neq j$ having attribute values of attributes $\widehat{h}^i, i = 1, 2, 3, \ldots, n, \widehat{h}^i \neq \widehat{h}^j, i \neq j$, respectively, and $\mathscr{W}$: $\mathscr{D} \longrightarrow \mathbb{P}(\mathscr{U})$ is called approximate function (so-called maa-function) of $\mathfrak{H}$, and for all $\widehat{d} \in \mathscr{D}, \mathscr{W}(\widehat{d})$ is called approximate element of $\mathfrak{H}$. The hs-set $\mathfrak{H}$ over $\mathscr{U}$ is said to be fhs-set and $ifhs$-set if $\mathscr{W}$: $\mathscr{D} \longrightarrow \mathbb{F}(\mathscr{U})$ and $\mathscr{W}$: $\mathscr{D} \longrightarrow \mathbb{IF}(\mathscr{U})$, respectively, where $\mathbb{F}(\mathscr{U})$ and $\mathbb{IF}(\mathscr{U})$ denote the family of all fuzzy subsets and intuitionistic fuzzy subsets, respectively.

In 2021, Chellamani et al. [32] explored the graphical notations of pfs-sets to handle pfs-information efficiently.

Definition 5 (see [32]). Let $\mathbb{M}^\star = (\mathfrak{B}, \mathbb{E})$ be a simple graph with $\mathfrak{B}$ as set of vertices and $\mathbb{E}$ as set of edges and $\mathfrak{E}$ be a nonvoid set of parameters. By a picture fuzzy soft graph (pfs-graph), we mean a four-tuple $\mathfrak{G} = (\mathfrak{G}^\star, \mathfrak{Q}, \mathbb{L}, \mathbb{M})$ with $(\mathbb{L}, \mathfrak{E})$ and $(\mathbb{M}, \mathfrak{E})$ that are pfs-sets over $\mathfrak{B}$ and $\mathbb{E}$, respectively; for all $\widehat{e} \in \mathfrak{E}$, $(\mathbb{L}(\widehat{e}), \mathbb{M}(\widehat{e}))$ is a pfs-graph of $\mathfrak{G}$ if $\mu_T^{\mathbb{M}(\widehat{e})}(\widehat{b}_1\widehat{b}_2) \leq \min\left\{\mu_T^{\mathbb{L}(\widehat{e})}(\widehat{b}_1), \mu_T^{\mathbb{L}(\widehat{e})}(\widehat{b}_2)\right\}$, $\mu_N^{\mathbb{M}(\widehat{e})}(\widehat{b}_1\widehat{b}_2) \leq \min\left\{\mu_N^{\mathbb{L}(\widehat{e})}(\widehat{b}_1), \mu_N^{\mathbb{L}(\widehat{e})}(\widehat{b}_2)\right\}$, and $\mu_F^{\mathbb{M}(\widehat{e})}(\widehat{b}_1\widehat{b}_2) \leq \max\left\{\mu_F^{\mathbb{L}(\widehat{e})}(\widehat{b}_1), \mu_F^{\mathbb{L}(\widehat{e})}(\widehat{b}_2)\right\}$ such that $0 \leq \mu_T^{\mathbb{M}(\widehat{e})}(\widehat{b}_1\widehat{b}) + \mu_N^{\mathbb{M}(\widehat{e})}(\widehat{b}_1\widehat{b}) + \mu_F^{\mathbb{M}(\widehat{e})}(\widehat{b}_1\widehat{b})_2 \leq 1$ for all $\widehat{b}_1, \widehat{b}_2 \in \mathfrak{B}$.

3. Possibility Picture Fuzzy Hypersoft Graphs

This portion presents the characterization of popfhs-graph along with some essential properties and results. First, we present the definitions of picture fuzzy hypersoft set $pfhs$-set and possibility picture fuzzy hypersoft set popfhs-set, which are missing in existing literature.

Definition 6. A $pfhs$-set $\mathbb{H}$ over $\mathscr{U}$ is a pair $(\pi_{\mathbb{H}}, \mathbb{Z})$, where $\pi_{\mathbb{H}}$: $\mathbb{Z} \longrightarrow \mathbb{P}_{\mathscr{F}}(\mathscr{U})$ and $\mathbb{Z} = \Pi_{i=1}^n \mathbb{Z}_i$ where $\mathbb{Z}_i \, (i = 1, 2, \ldots, n)$ is nonoverlapping attribute-valued sets with respect to distinct attributes. For any $\widehat{z} \in \mathbb{Z}, \pi_{\mathbb{H}}(\widehat{z})$ is a $pfhs$-subset and known as approximate element of $pfhs$-set $\mathbb{H}$, that is, $\pi_{\mathbb{H}}(\widehat{z})$ can be represented as $pfhs$-set over $\mathscr{U}$ such that $\pi_{\mathbb{H}}(\widehat{z}) = \left\{(\widehat{u}, < \mu_T^{\widehat{z}}(\widehat{u}), \mu_N^{\widehat{z}}(\widehat{u}), \mu_F^{\widehat{z}}(\widehat{u}) >)|\widehat{u} \in \mathscr{U}\right\}$ where $\mu_T^{\widehat{z}}(\widehat{u})$, $\mu_N^{\widehat{z}}(\widehat{u})$, and $\mu_F^{\widehat{z}}(\widehat{u})$ represent positive, neutral, and negative membership grades, respectively, of $\widehat{u} \in \mathscr{U}$ subject to the condition that $0 \leq \mu_T^{\widehat{z}}(\widehat{u}) + \mu_N^{\widehat{z}}(\widehat{u}) + \mu_F^{\widehat{z}}(\widehat{u}) \leq 1$ with refusal membership grade $\mu_R^{\widehat{z}}(\widehat{u}) = 1 - \mu_T^{\widehat{z}}(\widehat{u}) - \mu_N^{\widehat{z}}(\widehat{u}) - \mu_F^{\widehat{z}}(\widehat{u})$.

Definition 7. A possibility picture fuzzy hypersoft set (popfhs-set) $\mathbb{H}_p$ over $\mathscr{U}$ is stated as $\mathbb{H}_p = \{(\widehat{z}, < \pi_{\mathbb{H}}(\widehat{z}) >, \delta(\widehat{z}))|\widehat{z} \in \mathbb{Z}\}$, where $\pi_{\mathbb{H}}(\widehat{z})$ is a $pfhs$-set as defined in Definition 6 and δ: $\mathscr{U} \longrightarrow [0,1]$ with $\delta(\widehat{z})$ is the possibility degree of $\widehat{z}$ to $\mathbb{H}_p$. The collection of all $popfhs$-sets over $\mathscr{U}$ is represented by $\Omega_{\mathbb{PF}}(\mathscr{U})$.

Throughout the remaining paper, $\mathfrak{A}^\star = (\mathfrak{B}, \mathfrak{E})$ is a simple graph with $\mathfrak{B}$ as set of vertices and $\mathfrak{E}$ as set of edges; $\mathbb{E}$ denotes set of parameters; and $\mathfrak{Q}^i$ denotes the disjoint sets containing subparametric values for distinct parameters $\widehat{e}_i, i = 1, 2, \ldots, n$ of $\mathbb{E}$. Also, $\mathfrak{Q} = \mathfrak{Q}^1 \times \mathfrak{Q}^2 \times \mathfrak{Q}^3 \times \cdots \times \mathfrak{Q}^n$.

Definition 8. A popfhs-graph is a four-tuple $\mathfrak{A} = (\mathfrak{A}^\star, \mathfrak{Q}, \mathbb{J}, \mathbb{K})$, where $\mathbb{J}$: $\mathfrak{Q} \longrightarrow \mathbb{P}_F(\mathfrak{B})$, $\mathbb{K}$: $\mathfrak{Q} \longrightarrow \mathbb{P}_F(\mathfrak{B} \times \mathfrak{B})$ given by $\mathbb{J}(\sigma) = \mathbb{J}_\sigma = \left\{((\widehat{v}), \prec \mathbb{T}_{\mathbb{J}_\sigma}(\widehat{v}), \mathbb{I}_{\mathbb{J}_\sigma}(\widehat{v}), \mathbb{F}_{\mathbb{J}_\sigma}(\widehat{v}) \succ, \mu(\widehat{v})), \widehat{v} \in \mathfrak{B}\right\}$ and $\mathbb{K}(\sigma) = \mathbb{K}_\sigma = \left\{((\widehat{v}_1, \widehat{v}_2), \prec \mathbb{T}_{\mathbb{J}_\sigma}(\widehat{v}_1, \widehat{v}_2), \mathbb{I}_{\mathbb{J}_\sigma}(\widehat{v}_1, \widehat{v}_2), \mathbb{F}_{\mathbb{J}_\sigma}(\widehat{v}_1, \widehat{v}_2) \succ, \mu_{\mathbb{J}_\sigma}(\widehat{v}_1, \widehat{v}_2)), (\widehat{v}_1, \widehat{v}_2) \in \mathfrak{B} \times \mathfrak{B}\right\}$ are $popf$-sets over $\mathfrak{B}$ and $\mathfrak{B} \times \mathfrak{B}$ with $\mathbb{T}_{\mathbb{J}_\sigma}(\widehat{v}_1, \widehat{v}_2) \leq \min\left\{\mathbb{T}_{\mathbb{J}_\sigma}(\widehat{v}_1), \mathbb{T}_{\mathbb{J}_\sigma}(\widehat{v}_2)\right\}$, $\mathbb{I}_{\mathbb{J}_\sigma}(\widehat{v}_1, \widehat{v}_2) \leq \min\left\{\mathbb{I}_{\mathbb{J}_\sigma}(\widehat{v}_1), \mathbb{I}_{\mathbb{J}_\sigma}(\widehat{v}_2)\right\}$, $\mathbb{F}_{\mathbb{J}_\sigma}(\widehat{v}_1, \widehat{v}_2) \leq \max\left\{\mathbb{F}_{\mathbb{J}_\sigma}(\widehat{v}_1), \mathbb{F}_{\mathbb{J}_\sigma}(\widehat{v}_2)\right\}$, $\mu_{\mathbb{J}_\sigma}(\widehat{v}_1, \widehat{v}_2) \leq \min\left\{\mu_{\mathbb{J}_\sigma}(\widehat{v}_1), \mu_{\mathbb{J}_\sigma}(-\widehat{v}_2)\right\}$, $\forall \, (\widehat{v}_1, \widehat{v}_2) \in (\mathfrak{B} \times \mathfrak{B})$, and $\sigma \in \mathfrak{Q}$.

Note: The collection of all popfhs-graphs is represented by Ω_{PPFHSG}.

Example 1. Let $\mathfrak{A}^\star = (\mathfrak{B}, \mathfrak{E})$ be a simple graph with $\mathfrak{B} = \{\hat{v}_1, \hat{v}_2, \hat{v}_3\}$ and $\mathfrak{Q}_1 = \{\hat{q}_{11}, \hat{q}_{12}\}$, $\mathfrak{Q}_2 = \{\hat{q}_{21}, \hat{q}_{22}\}$, and $\mathfrak{Q}_3 = \{\hat{q}_{31}\}$ such that $\mathfrak{Q} = \mathfrak{Q}_1 \times \mathfrak{Q}_2 \times \mathfrak{Q}_3 = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ and $\mathbb{T}_{\mathbb{K}_\sigma}(\hat{v}_i, \hat{v}_j) = 0, \mathbb{I}_{\mathbb{K}_\sigma}(\hat{v}_i, \hat{v}_j) = 0, \mathbb{F}_{\mathbb{K}_\sigma}(\hat{v}_i, \hat{v}_j) = 1, \mu_{\mathbb{K}_\sigma}(\hat{v}_i, \hat{v}_j) = 0\, \forall (\hat{v}_i, \hat{v}_j) \in \mathfrak{B} \times \mathfrak{B}/\{(\hat{v}_1, \hat{v}_2), (\hat{v}_1, \hat{v}_2), (\hat{v}_1, \hat{v}_2)\}$. Table 1 and Figure 1 present its numerical tabulation and geometrical depiction, respectively.

Definition 9. A popfhs-graph $\mathfrak{G} = (\mathfrak{A}^\star, \mathfrak{Q}^\star, \mathbb{J}^1, \mathbb{K}^1)$ is called a popfhs-subgraph of $\mathfrak{A} = (\mathfrak{A}^\star, \mathfrak{Q}, \mathbb{J}, \mathbb{K})$ if

(1) $\mathfrak{Q}^\star \subseteq \mathfrak{Q}$

(2) $\mathbb{J}^1_\sigma \subseteq \mathbb{J}$ implies $\mathbb{T}_{\mathbb{J}^1_\sigma}(\hat{v}) \leq \mathbb{T}_{\mathbb{J}_\sigma}(\hat{v}), \mathbb{I}_{\mathbb{J}^1_\sigma}(\hat{v}) \leq \mathbb{I}_{\mathbb{J}_\sigma}(\hat{v}), \mathbb{F}_{\mathbb{J}^1_\sigma}(\hat{v}) \geq \mathbb{F}_{\mathbb{J}_\sigma}(\hat{v}), \mu_{\mathbb{J}^1_\sigma}(\hat{v}) \leq \mu_{\mathbb{J}_\sigma}(\hat{v})$

(3) $\mathbb{K}^1_\sigma \subseteq \mathbb{K}$ implies $\mathbb{T}_{\mathbb{K}^1_\sigma}(\hat{v}) \leq \mathbb{T}_{\mathbb{K}_\sigma}(\hat{v}), \mathbb{I}_{\mathbb{K}^1_\sigma}(\hat{v}) \leq \mathbb{I}_{\mathbb{K}_\sigma}(\hat{v}), \mathbb{F}_{\mathbb{K}^1_\sigma}(\hat{v}) \geq \mathbb{F}_{\mathbb{K}_\sigma}(\hat{v}), \mu_{\mathbb{K}^1_\sigma}(\hat{v}) \leq \mu_{\mathbb{K}_\sigma}(\hat{v}) \forall\, \sigma \in \mathfrak{Q}^\star$

Example 2. Repeating Example 1 with $\mathfrak{Q}_1 = \{\alpha_{11}, \alpha_{12}\}$, $\mathfrak{Q}_2 = \{\alpha_{21}\}$, and $\mathfrak{Q}_3 = \{\alpha_{31}\}$ and $\mathfrak{Q}^\star = \mathfrak{Q}_1 \times \mathfrak{Q}_2 \times \mathfrak{Q}_3 = \{\sigma_1, \sigma_2, \sigma_3\}$, we have a new popfhs-graph $\mathfrak{A} = (\mathfrak{A}^\star, \mathfrak{Q}^\star, \mathbb{J}^1, \mathbb{K}^1)$ which is popfhs-subgraph of popfhs-graph given in Example 1. Its tabular and geometrical depiction are given in Table 2 and Figure 2, respectively.

Definition 10. A popfhs-subgraph $(\mathfrak{A}^\star, \mathfrak{Q}^\star, \mathbb{J}^1, \mathbb{K}^1)$ is called a popfhs-spanning subgraph of popfhs-graph $(\mathfrak{A}^\star, \mathfrak{Q}, \mathbb{J}, \mathbb{K})$ when $\mathbb{J}^1_\sigma(\hat{v}) = \mathbb{J}_\sigma(\hat{v}) \forall \hat{v} \in \mathfrak{B}, \sigma \in \mathfrak{Q}$.

Definition 11. A popfhs-subgraph $(\mathfrak{A}^\star, \mathfrak{Q}^\star, \mathbb{J}^1, \mathbb{K}^1)$ is called a strong popfhs-subgraph (*SPFHS*-subgraph) of popfhs-graph $(\mathfrak{A}^\star, \mathfrak{Q}, \mathbb{J}, \mathbb{K})$ when $\mathbb{K}_\sigma(\hat{v}_1, \hat{v}_2) = \mathbb{J}_\sigma(\hat{v}_1) \cap \mathbb{J}_\sigma(\hat{v}_2)$ for $\hat{v}_1, \hat{v}_2 \in \mathfrak{B}$ and $\sigma \in \mathfrak{Q}$.

3.1. Set-Theoretic Operations of $popfhs$-Graphs. This segment of the paper investigates the basic set theoretic operations of popfhs-graphs along with their geometrical interpretations.

Definition 12. The union of two popfhs-graphs $\mathfrak{A}_1 = (\mathfrak{A}^\star_1, \mathfrak{Q}^1, \mathbb{J}^1, \mathbb{K}^1)$ and $\mathfrak{A}_2 = (\mathfrak{A}^\star_2, \mathfrak{Q}^2, \mathbb{J}^2, \mathbb{K}^2)$, denoted by $\mathfrak{A}_1 \cup \mathfrak{A}_2$, is a popfhs-graph $\mathfrak{A} = (\mathfrak{A}^\star, \mathfrak{Q}, \mathbb{J}, \mathbb{K})$ such that $\mathfrak{Q} = \mathfrak{Q}^1 \cup \mathfrak{Q}^2$. In this graph,

Table 1: Numerical computation of Example 1 with (a) $\mathbb{P}_F(\sigma_1)$, (b) $\mathbb{P}_F(\sigma_2)$, (c) $\mathbb{P}_F(\sigma_3)$, and (d) $\mathbb{P}_F(\sigma_4)$

$\mathbb{J}$	$\hat{v}_1$	$\hat{v}_2$	$\hat{v}_3$
σ_1	(0.2, 0.1, 0.3, 0.2)	(0.1, 0.3, 0.2, 0.3)	(0, 0, 1, 0)
σ_2	(0.1, 0.3, 0.3, 0.2)	(0.2, 0.4, 0.1, 0.1)	(0, 0, 1, 0)
σ_3	(0.1, 0.2, 0.2, 0.3)	(0.3, 0.1, 0.2, 0.2)	(0.1, 0.2, 0.3, 0.1)
σ_4	(0.2, 0.2, 0.1, 0.3)	(0.3, 0.2, 0.1, 0.4)	(0.1, 0.3, 0.2, 0.2)
$\mathbb{K}$	$(\hat{v}_1, \hat{v}_2)$	$(\hat{v}_2, \hat{v}_3)$	$(\hat{v}_1, \hat{v}_3)$
σ_1	(0, 0, 1, 0)	(0, 0, 1, 0)	(0, 0, 1, 0)
σ_2	(0.1, 0.2, 0.2, 0.4)	(0, 0, 1, 0)	(0, 0, 1, 0)
σ_3	(0.1, 0.3, 0.3, 0.3)	(0.2, 0.1, 0.3, 0.1)	(0, 0, 1, 0)
σ_4	(0.1, 0.1, 0.1, 0.6)	(0.2, 0.2, 0.1, 0.4)	(0.2, 0.2, 0.1, 0.5)

$$\mathbb{T}_{\mathbb{J}_\sigma}(\hat{v}) = \begin{cases} \mathbb{T}_{\mathbb{J}^1_\sigma}(\hat{v}) \begin{cases} \text{if } \sigma \in \mathfrak{Q}^1 - \mathfrak{Q}^2 \& \hat{v} \in \mathfrak{B}_1 - \mathfrak{B}_2 \\ \text{or if } \sigma \in \mathfrak{Q}^1 - \mathfrak{Q}^2 \& \hat{v} \in \mathfrak{B}_1 \cap \mathfrak{B}_2 \\ \text{or if } \sigma \in \mathfrak{Q}^1 \cap \mathfrak{Q}^2 \& \hat{v} \in \mathfrak{B}_1 - \mathfrak{B}_2 \end{cases} \\ \mathbb{T}_{\mathbb{J}^2_\sigma}(\hat{v}) \begin{cases} \text{if } \sigma \in \mathfrak{Q}^2 - \mathfrak{Q}^1 \& \hat{v} \in \mathfrak{B}_2 - \mathfrak{B}_1 \\ \text{or if } \sigma \in \mathfrak{Q}^2 - \mathfrak{Q}^1 \& \hat{v} \in \mathfrak{B}_2 \cap \mathfrak{B}_1 \\ \text{or if } \sigma \in \mathfrak{Q}^1 \cap \mathfrak{Q}^2 \& \hat{v} \in \mathfrak{B}_2 - \mathfrak{B}_1 \end{cases} \\ \max\left\{\mathbb{T}_{\mathbb{J}^1_\sigma}(\hat{v}), \mathbb{T}_{\mathbb{J}^2_\sigma}(\hat{v})\right\}\left\{\text{if } \sigma \in \mathfrak{Q}^1 \cap \mathfrak{Q}^2 \& \hat{v} \in \mathfrak{B}_1 \cap \mathfrak{B}_2\right. \\ 0, \text{otherwise} \end{cases}$$

$$\mathbb{I}_{\mathbb{J}_\sigma}(\hat{v}) = \begin{cases} \mathbb{I}_{\mathbb{J}^1_\sigma}(\hat{v}) \begin{cases} \text{if } \sigma \in \mathfrak{Q}^1 - \mathfrak{Q}^2 \& \hat{v} \in \mathfrak{B}_1 - \mathfrak{B}_2 \\ \text{or if } \sigma \in \mathfrak{Q}^1 - \mathfrak{Q}^2 \hat{v} \in \mathfrak{B}_1 \cap \mathfrak{B}_2 \\ \text{or if } \sigma \in \mathfrak{Q}^1 \cap \mathfrak{Q}^2 \& \hat{v} \in \mathfrak{B}_1 - \mathfrak{B}_2 \end{cases} \\ \mathbb{I}_{\mathbb{J}^2_\sigma}(\hat{v}) \begin{cases} \text{if } \sigma \in \mathfrak{Q}^2 - \mathfrak{Q}^1 \& \hat{v} \in \mathfrak{B}_2 - \mathfrak{B}_1 \\ \text{or if } \sigma \in \mathfrak{Q}^2 - \mathfrak{Q}^1 \& \hat{v} \in \mathfrak{B}_2 \cap \mathfrak{B}_1 \\ \text{or if } \sigma \in \mathfrak{Q}^1 \cap \mathfrak{Q}^2 \& \hat{v} \in \mathfrak{B}_2 - \mathfrak{B}_1 \end{cases} \\ \max\left\{\mathbb{I}_{\mathbb{J}^1_\sigma}(\hat{v}), \mathbb{I}_{\mathbb{J}^2_\sigma}(\hat{v})\right\} \text{if } \sigma \in \mathfrak{Q}^1 \cap \mathfrak{Q}^2 \& \hat{v} \in \mathfrak{B}_1 \cap \mathfrak{B}_2 \\ 0, \text{otherwise} \end{cases} \&$$

$$\mathbb{F}_{\mathbb{J}_\sigma}(\hat{v}) = \begin{cases} \mathbb{F}_{\mathbb{J}^1_\sigma}(\hat{v}) \begin{cases} \text{if } \sigma \in \mathfrak{Q}^1 - \mathfrak{Q}^2 \& \hat{v} \in \mathfrak{B}_1 - \mathfrak{B}_2 \\ \text{or if } \sigma \in \mathfrak{Q}^1 - \mathfrak{Q}^2 \& \hat{v} \in \mathfrak{B}_1 \cap \mathfrak{B}_2 \\ \text{or if } \sigma \in \mathfrak{Q}^1 \cap \mathfrak{Q}^2 \& \hat{v} \in \mathfrak{B}_1 - \mathfrak{B}_2 \end{cases} \\ \mathbb{F}_{\mathbb{J}^2_\sigma}(\hat{v}) \begin{cases} \text{if } \sigma \in \mathfrak{Q}^2 - \mathfrak{Q}^1 \& \hat{v} \in \mathfrak{B}_2 - \mathfrak{B}_1 \\ \text{or if } \sigma \in \mathfrak{Q}^2 - \mathfrak{Q}^1 \& \hat{v} \in \mathfrak{B}_2 \cap \mathfrak{B}_1 \\ \text{or if } \sigma \in \mathfrak{Q}^1 \cap \mathfrak{Q}^2 \& \hat{v} \in \mathfrak{B}_2 - \mathfrak{B}_1 \end{cases} \\ \min\left\{\mathbb{F}_{\mathbb{J}^1_\sigma}(\hat{v}), \mathbb{F}_{\mathbb{J}^2_\sigma}(\hat{v})\right\}\left\{\text{if } \sigma \in \mathfrak{Q}^1 \cap \mathfrak{Q}^2 \& \hat{v} \in \mathfrak{B}_1 \cap \mathfrak{B}_2\right. \\ 0, \text{otherwise.} \end{cases} \& \tag{1}$$

Also, the *popfhs*-components for $\mathbb{K}$ are given as follows:

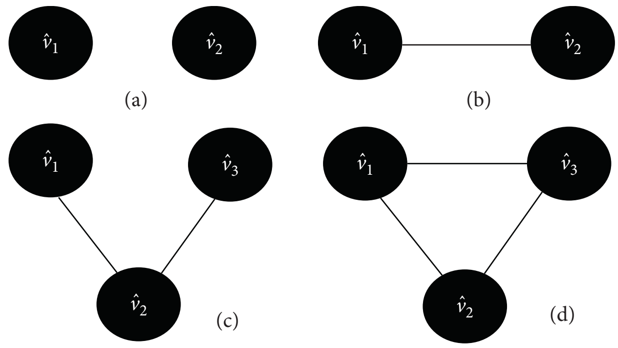


Figure 1: Geometrical interpretation of Table 1.

Table 2: Tabular notation of Example 2 with (a) $\mathbb{P}_F(\sigma_1)$, (b) $\mathbb{P}_F(\sigma_2)$, and (c) $\mathbb{P}_F(\sigma_3)$.

$\mathbb{J}$	$\hat{v}_1$	$\hat{v}_2$	$\hat{v}_3$
σ_1	$(0.1, 0.3, 0.1, 0.1)$	$(0.2, 0.2, 0.1, 0.2)$	$(0, 0, 1, 0)$
σ_2	$(0.1, 0.2, 0.1, 0.1)$	$(0.2, 0.1, 0.3, 0.1)$	$(0, 0, 1, 0)$
σ_3	$(0.1, 0.4, 0.1, 0.2)$	$(0.2, 0.2, 0.1, 0.1)$	$(0.2, 0.1, 0.2, 0.1)$
$\mathbb{K}$	$(\hat{v}_1, \hat{v}_2)$	$(\hat{v}_2, \hat{v}_3)$	$(\hat{v}_1, \hat{v}_3)$
σ_1	$(0, 0, 1, 0)$	$(0, 0, 1, 0)$	$(0, 0, 1, 0)$
σ_2	$(0.1, 0.2, 0.1, 0.1)$	$(0, 0, 1, 0)$	$(0, 0, 1, 0)$
σ_3	$(0.1, 0.2, 0.3, 0.2)$	$(0.1, 0.2, 0.4, 0.1)$	$(0, 0, 1, 0)$

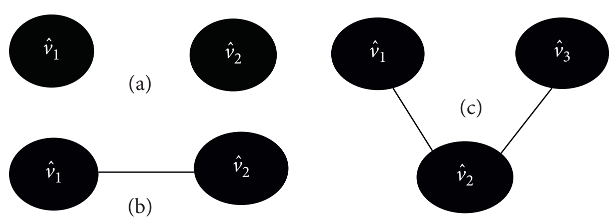


Figure 2: Graphical representation of Table 2.

$$\mathbb{T}_{\mathbb{K}_\sigma}(\hat{v}_1) = \begin{cases} \mathbb{T}_{\mathbb{K}_\sigma^1}(\hat{v}_1) \begin{cases} \text{if } \sigma \in \mathfrak{Q}^1 - \mathfrak{Q}^2 \& (\hat{v}_1, \hat{v}_2) \in (\mathfrak{B}_1 \times \mathfrak{B}_1) - (\mathfrak{B}_2 \times \mathfrak{B}_2) \\ \text{or if } \sigma \in \mathfrak{Q}^1 - \mathfrak{Q}^2 \& (\hat{v}_1, \hat{v}_2) \in (\mathfrak{B}_1 \times \mathfrak{B}_1) \cap (\mathfrak{B}_2 \times \mathfrak{B}_2) \\ \text{or if } \sigma \in \mathfrak{Q}^1 \cap \mathfrak{Q}^2 \& (\hat{v}_1, \hat{v}_2) \in (\mathfrak{B}_1 \times \mathfrak{B}_1) - (\mathfrak{B}_2 \times \mathfrak{B}_2), \end{cases} \\ \mathbb{T}_{\mathbb{K}_\sigma^2}(\hat{v}_1) \begin{cases} \text{if } \sigma \in \mathfrak{Q}^2 - \mathfrak{Q}^1 \& (\hat{v}_1, \hat{v}_2) \in (\mathfrak{B}_2 \times \mathfrak{B}_2) - (\mathfrak{B}_1 \times \mathfrak{B}_1) \\ \text{or if } \sigma \in \mathfrak{Q}^2 - \mathfrak{Q}^1 \& (\hat{v}_1, \hat{v}_2) \in (\mathfrak{B}_2 \times \mathfrak{B}_2) \cap (\mathfrak{B}_1 \times \mathfrak{B}_1) \\ \text{or if } \sigma \in \mathfrak{Q}^1 \cap \mathfrak{Q}^2 \& (\hat{v}_1, \hat{v}_2) \in (\mathfrak{B}_2 \times \mathfrak{B}_2) - (\mathfrak{B}_1 \times \mathfrak{B}_1), \end{cases} \\ \max\left\{\mathbb{T}_{\mathbb{K}_\sigma^1}(\hat{v}_1), \mathbb{T}_{\mathbb{K}_\sigma^2}(\hat{v}_1)\right\}\left\{if\ \sigma \in \mathfrak{Q}^1 \cap \mathfrak{Q}^2 \& (\hat{v}_1, \hat{v}_2) \in (\mathfrak{B}_1 \times \mathfrak{B}_1) \cap (\mathfrak{B}_2 \times \mathfrak{B}_2),\right. \\ 0, \text{ otherwise,} \end{cases}$$

$$
\mathbb{I}_{\mathbb{K}_\sigma}(\hat{v}_1)=\begin{cases}
\mathbb{I}_{\mathbb{K}^1_\sigma}(\hat{v}_1)\begin{cases}\text{if } \sigma\in\mathfrak{Q}^1-\mathfrak{Q}^2\,\&\,(\hat{v}_1,\hat{v}_2)\in(\mathfrak{B}_1\times\mathfrak{B}_1)-(\mathfrak{B}_2\times\mathfrak{B}_2)\\ \text{or if } \sigma\in\mathfrak{Q}^1-\mathfrak{Q}^2\,\&\,(\hat{v}_1,\hat{v}_2)\in(\mathfrak{B}_1\times\mathfrak{B}_1)\cap(\mathfrak{B}_2\times\mathfrak{B}_2)\\ \text{or if } \sigma\in\mathfrak{Q}^1\cap\mathfrak{Q}^2\,\&\,(\hat{v}_1,\hat{v}_2)\in(\mathfrak{B}_1\times\mathfrak{B}_1)-(\mathfrak{B}_2\times\mathfrak{B}_2),\end{cases}\\
\mathbb{I}_{\mathbb{K}^2_\sigma}(\hat{v}_1)\begin{cases}\text{if } \sigma\in\mathfrak{Q}^2-\mathfrak{Q}^1\,\&\,(\hat{v}_1,\hat{v}_2)\in(\mathfrak{B}_2\times\mathfrak{B}_2)-(\mathfrak{B}_1\times\mathfrak{B}_1)\\ \text{or if } \sigma\in\mathfrak{Q}^2-\mathfrak{Q}^1\,\&\,(\hat{v}_1,\hat{v}_2)\in(\mathfrak{B}_2\times\mathfrak{B}_2)\cap(\mathfrak{B}_1\times\mathfrak{B}_1)\\ \text{or if } \sigma\in\mathfrak{Q}^1\cap\mathfrak{Q}^2\,\&\,(\hat{v}_1,\hat{v}_2)\in(\mathfrak{B}_2\times\mathfrak{B}_2)-(\mathfrak{B}_1\times\mathfrak{B}_1),\end{cases}\\
\max\left\{\mathbb{I}_{\mathbb{K}^1_\sigma}(\hat{v}_1),\mathbb{I}_{\mathbb{K}^2_\sigma}(\hat{v}_1)\right\}\left\{\text{if } \sigma\in\mathfrak{Q}^1\cap\mathfrak{Q}^2\,\&\,(\hat{v}_1,\hat{v}_2)\in(\mathfrak{B}_1\times\mathfrak{B}_1)\cap(\mathfrak{B}_2\times\mathfrak{B}_2),\right.\\
0,\ \text{otherwise},
\end{cases}
$$

$$
\mathbb{F}_{\mathbb{K}_\sigma}(\hat{v}_1)=\begin{cases}
\mathbb{F}_{\mathbb{K}^1_\sigma}(\hat{v}_1)\begin{cases}\text{if } \sigma\in\mathfrak{Q}^1-\mathfrak{Q}^2\,\&\,(\hat{v}_1,\hat{v}_2)\in(\mathfrak{B}_1\times\mathfrak{B}_1)-(\mathfrak{B}_2\times\mathfrak{B}_2)\\ \text{or if } \sigma\in\mathfrak{Q}^1-\mathfrak{Q}^2\,\&\,(\hat{v}_1,\hat{v}_2)\in(\mathfrak{B}_1\times\mathfrak{B}_1)\cap(\mathfrak{B}_2\times\mathfrak{B}_2)\\ \text{or if } \sigma\in\mathfrak{Q}^1\cap\mathfrak{Q}^2\,\&\,(\hat{v}_1,\hat{v}_2)\in(\mathfrak{B}_1\times\mathfrak{B}_1)-(\mathfrak{B}_2\times\mathfrak{B}_2),\end{cases}\\
\mathbb{F}_{\mathbb{K}^2_\sigma}(\hat{v}_1)\begin{cases}\text{if } \sigma\in\mathfrak{Q}^2-\mathfrak{Q}^1\,\&\,(\hat{v}_1,\hat{v}_2)\in(\mathfrak{B}_2\times\mathfrak{B}_2)-(\mathfrak{B}_1\times\mathfrak{B}_1)\\ \text{or if } \sigma\in\mathfrak{Q}^2-\mathfrak{Q}^1\,\&\,(\hat{v}_1,\hat{v}_2)\in(\mathfrak{B}_2\times\mathfrak{B}_2)\cap(\mathfrak{B}_1\times\mathfrak{B}_1)\\ \text{or if } \sigma\in\mathfrak{Q}^1\cap\mathfrak{Q}^2\,\&\,(\hat{v}_1,\hat{v}_2)\in(\mathfrak{B}_2\times\mathfrak{B}_2)-(\mathfrak{B}_1\times\mathfrak{B}_1),\end{cases}\\
\min\left\{\mathbb{F}_{\mathbb{K}^1_\sigma}(\hat{v}_1),\mathbb{F}_{\mathbb{K}^2_\sigma}(\hat{v}_1)\right\}\left\{\text{if } \sigma\in\mathfrak{Q}^1\cap\mathfrak{Q}^2\,\&\,(\hat{v}_1,\hat{v}_2)\in(\mathfrak{B}_1\times\mathfrak{B}_1)\cap(\mathfrak{B}_2\times\mathfrak{B}_2),\right.\\
0,\ \text{otherwise},
\end{cases}\tag{2}
$$

$$
\mu_{\mathbb{K}_\sigma}(\hat{v}_1)=\begin{cases}
\mu_{\mathbb{K}^1_\sigma}(\hat{v}_1)\begin{cases}\text{if } \sigma\in\mathfrak{Q}^1-\mathfrak{Q}^2\,\&\,(\hat{v}_1,\hat{v}_2)\in(\mathfrak{B}_1\times\mathfrak{B}_1)-(\mathfrak{B}_2\times\mathfrak{B}_2)\\ \text{or if } \sigma\in\mathfrak{Q}^1-\mathfrak{Q}^2\,\&\,(\hat{v}_1,\hat{v}_2)\in(\mathfrak{B}_1\times\mathfrak{B}_1)\cap(\mathfrak{B}_2\times\mathfrak{B}_2)\\ \text{or if } \sigma\in\mathfrak{Q}^1\cap\mathfrak{Q}^2\,\&\,(\hat{v}_1,\hat{v}_2)\in(\mathfrak{B}_1\times\mathfrak{B}_1)-(\mathfrak{B}_2\times\mathfrak{B}_2),\end{cases}\\
\mu_{\mathbb{K}^2_\sigma}(\hat{v}_1)\begin{cases}\text{if } \sigma\in\mathfrak{Q}^2-\mathfrak{Q}^1\,\&\,(\hat{v}_1,\hat{v}_2)\in(\mathfrak{B}_2\times\mathfrak{B}_2)-(\mathfrak{B}_1\times\mathfrak{B}_1)\\ \text{or if } \sigma\in\mathfrak{Q}^2-\mathfrak{Q}^1\,\&\,(\hat{v}_1,\hat{v}_2)\in(\mathfrak{B}_2\times\mathfrak{B}_2)\cap(\mathfrak{B}_1\times\mathfrak{B}_1)\\ \text{or if } \sigma\in\mathfrak{Q}^1\cap\mathfrak{Q}^2\,\&\,(\hat{v}_1,\hat{v}_2)\in(\mathfrak{B}_2\times\mathfrak{B}_2)-(\mathfrak{B}_1\times\mathfrak{B}_1),\end{cases}\\
\max\left\{\mu_{\mathbb{K}^1_\sigma}(\hat{v}_1),\mu_{\mathbb{K}^2_\sigma}(\hat{v}_1)\right\}\left\{\text{if } \sigma\in\mathfrak{Q}^1\cap\mathfrak{Q}^2\,\&\,(\hat{v}_1,\hat{v}_2)\in(\mathfrak{B}_1\times\mathfrak{B}_1)\cap(\mathfrak{B}_2\times\mathfrak{B}_2),\right.\\
0,\ \text{otherwise}.
\end{cases}
$$

Theorem 1. *If* $\mathfrak{A}_1,\mathfrak{A}_2\in\Omega_{PFHSG}$, *then* $\mathfrak{A}_1\cup\mathfrak{A}_2\in\Omega_{PFHSG}$.

Proof. Its proof is trivial in accordance with Definition 12. □

Example 3. Let $\mathfrak{A}_1=(\mathfrak{A}_1^\star,\mathfrak{Q}^1,\mathbb{J}^1,\mathbb{K}^1)$ be a popfhs-graph where $\mathfrak{A}_1^\star=(\mathfrak{B}_1,\mathfrak{E}_1)$ with $\mathfrak{B}_1=\{\hat{v}_1,\hat{v}_2,\hat{v}_3\}$ and $\mathfrak{Q}_1=\{\alpha_{11}\}$, $\mathfrak{Q}_2=\{\alpha_{21}\}$, and $\mathfrak{Q}_3=\{\alpha_{31},\alpha_{32},\alpha_{33}\}$ such that $\mathfrak{Q}^1=\mathfrak{Q}_1\times\mathfrak{Q}_2\times\mathfrak{Q}_3=\{\sigma_1,\sigma_2,\sigma_3\}$ and $\mathbb{T}_{\mathbb{K}_\sigma}(\hat{v}_i,\hat{v}_j)=0,\mathbb{I}_{\mathbb{K}_\sigma}(\hat{v}_i,\hat{v}_j)=0,\mathbb{F}_{\mathbb{K}_\sigma}(\hat{v}_i,\hat{v}_j)=1,\mu_{\mathbb{K}_\sigma}(\hat{v}_i,\hat{v}_j)=0$, and $(\forall(\hat{v}_i,\hat{v}_j)\in\mathfrak{B}_1\times\mathfrak{B}_1/\{(\hat{v}_1,\hat{v}_2),(\hat{v}_2,\hat{v}_3),(\hat{v}_1,\hat{v}_3)\})$. Its tabulation is given in Table 3. Also, let $\mathfrak{A}_2=(\mathfrak{A}_2^\star,\mathfrak{Q}^2,\mathbb{J}^2,\mathbb{K}^2)$ be a popfhs-graph, where $\mathfrak{A}_2^\star=(\mathfrak{B}_2,\mathfrak{E}_2)$ with $\mathfrak{B}_2=\{\hat{v}_3,\hat{v}_4,\hat{v}_5\}$, $\mathfrak{Q}_3=\{\alpha_{31},\alpha_{32}\}$, and $\mathfrak{Q}_4=\{\alpha_{41}\}$ such that $\mathfrak{Q}_5=\{\alpha_{51}\}$. $\mathfrak{Q}^2=\mathfrak{Q}_3\times\mathfrak{Q}_4\times\mathfrak{Q}_5=\{\sigma_2,\sigma_4\}$ and $\mathbb{T}_{\mathbb{K}_\sigma}(\hat{v}_i,\hat{v}_j)=0,\mathbb{I}_{\mathbb{K}_\sigma}(\hat{v}_i,\hat{v}_j)=0,\mathbb{F}_{\mathbb{K}_\sigma}(\hat{v}_i,\hat{v}_j)=1$, $\mu_{\mathbb{K}_\sigma}(\hat{v}_i,\hat{v}_j)=0$, and $(\forall(\hat{v}_i,\hat{v}_j)\in\mathfrak{B}_2\times\mathfrak{B}_2/\{(\hat{v}_3,\hat{v}_4),(\hat{v}_4,\hat{v}_5),(\hat{v}_3,\hat{v}_5)\})$. Its tabulation is given in Table 4. Now let $\mathfrak{A}=\mathfrak{A}_1\cup\mathfrak{A}_2$ with $\mathfrak{Q}=\mathfrak{Q}^1\cup\mathfrak{Q}^2$ and $\mathbb{T}_{\mathbb{K}_\sigma}(\hat{v}_i,\hat{v}_j)=0,\mathbb{I}_{\mathbb{K}_\sigma}(\hat{v}_i,\hat{v}_j)=0$, $\mathbb{F}_{\mathbb{K}_\sigma}(\hat{v}_i,\hat{v}_j)=1,\mu_{\mathbb{K}_\sigma}(\hat{v}_i,\hat{v}_j)=0$, and $(\forall(\hat{v}_i,\hat{v}_j)\in\mathfrak{B}\times\mathfrak{B}/\{(\hat{v}_1,\hat{v}_2),(\hat{v}_1,\hat{v}_3),(\hat{v}_2,\hat{v}_3),(\hat{v}_3,\hat{v}_4),(\hat{v}_3,\hat{v}_5),(\hat{v}_4,\hat{v}_5)\})$. Its tabulation is given in Table 5.

The graphical interpretations of Tables 3–5 are presented in Figures 3–5, respectively.

Definition 13. The intersection of two popfhs-graphs $\mathfrak{G}_1=(\mathfrak{G}_1^\star,\mathfrak{Q}^1,\mathbb{J}^1,\mathbb{K}^1)$ and $\mathfrak{G}_2=(\mathfrak{G}_2^\star,\mathfrak{Q}^2,\mathbb{J}^2,\mathbb{K}^2)$, denoted by $\mathfrak{G}_1\cap\mathfrak{G}_2$, is a popfhs-graph $G=(\mathfrak{G}^\star,\mathfrak{Q},\mathbb{J},\mathbb{K})$ such that $\mathfrak{Q}=\mathfrak{Q}^1\cap\mathfrak{Q}^2,\mathfrak{B}=\mathfrak{B}_1\cap\mathfrak{B}_2$. In this graph, the uncertain parts for $\mathbb{J}$ are as follows:

Table 3: Tabulation of Example 3.

$\mathbb{J}$	$\widehat{v}_1$	$\widehat{v}_2$	$\widehat{v}_3$
σ_1	$(0.2, 0.1, 0.2, 0.2)$	$(0.3, 0.1, 0.1, 0.3)$	$(0.2, 0.2, 0.1, 0.3)$
σ_2	$(0.2, 0.3, 0.1, 0.2)$	$(0.2, 0.2, 0.1, 0.2)$	$(0.1, 0.2, 0.1, 0.5)$
σ_3	$(0.2, 0.1, 0.1, 0.6)$	$(0.1, 0.1, 0.4, 0.4)$	$(0.1, 0.2, 0.1, 0.7)$
$\mathbb{K}$	$(\widehat{v}_1, \widehat{v}_2)$	$(\widehat{v}_2, \widehat{v}_3)$	$(\widehat{v}_1, \widehat{v}_3)$
σ_1	$(0.2, 0.3, 0.2, 0.2)$	$(0.2, 0.1, 0.3, 0.2)$	$(0.2, 0.1, 0.2, 0.2)$
σ_2	$(0.2, 0.1, 0.3, 0.2)$	$(0.2, 0.2, 0.1, 0.2)$	$(0.2, 0.3, 0.1, 0.2)$
σ_3	$(0, 0, 1, 0)$	$(0.1, 0.2, 0.1, 0.3)$	$(0.1, 0.2, 0.1, 0.2)$

Table 4: Tabulation of popfhs-graph $\mathfrak{A}_2 = (\mathfrak{A}_2^{\star}, \mathfrak{Q}^2, \mathbb{J}^2, \mathbb{K}^2)$ according to Example 3.

$\mathbb{J}$	$\widehat{v}_3$	$\widehat{v}_4$	$\widehat{v}_5$
σ_2	$(0.2, 0.1, 0.2, 0.3)$	$(0.2, 0.1, 0.2, 0.2)$	$(0.1, 0.2, 0.1, 0.5)$
σ_4	$(0.2, 0.1, 0.1, 0.6)$	$(0.1, 0.2, 0.1, 0.4)$	$(0.2, 0.1, 0.2, 0.4)$
$\mathbb{K}$	$(\widehat{v}_3, \widehat{v}_4)$	$(\widehat{v}_4, \widehat{v}_5)$	$(\widehat{v}_3, \widehat{v}_5)$
σ_2	$(0.2, 0.3, 0.1, 0.2)$	$(0.3, 0.1, 0.2, 0.3)$	$(0, 0, 1, 0)$
σ_4	$(0.2, 0.1, 0.3, 0.2)$	$(0.1, 0.3, 0.2, 0.3)$	$(0.1, 0.2, 0.2, 0.3)$

Table 5: Tabulation of $\mathfrak{A} = \mathfrak{A}_1 \cup \mathfrak{A}_2$.

$\mathbb{J}$	$\widehat{v}_1$	$\widehat{v}_2$	$\widehat{v}_3$	$\widehat{v}_4$	$\widehat{v}_5$	
σ_1	$(0.2, 0.1, 0.2, 0.2)$	$(0.3, 0.1, 0.1, 0.3)$	$(0.2, 0.2, 0.1, 0.3)$	$(0, 0, 1, 0)$	$(0, 0, 1, 0)$	
σ_2	$(0.2, 0.3, 0.1, 0.2)$	$(0.2, 0.2, 0.1, 0.2)$	$(0.2, 0.2, 0.1, 0.5)$	$(0.2, 0.1, 0.2, 0.2)$	$(0.1, 0.2, 0.1, 0.5)$	
σ_3	$(0.2, 0.1, 0.1, 0.6)$	$(0.1, 0.1, 0.4, 0.4)$	$(0.1, 0.2, 0.1, 0.7)$	$(0, 0, 1, 0)$	$(0, 0, 1, 0)$	
σ_4	$(0, 0, 1, 0)$	$(0, 0, 1, 0)$	$(0.2, 0.1, 0.1, 0.6)$	$(0.1, 0.2, 0.1, 0.4)$	$(0.2, 0.1, 0.2, 0.4)$	
$\mathbb{K}$	$(\widehat{v}_1, \widehat{v}_2)$	$(\widehat{v}_1, \widehat{v}_3)$	$(\widehat{v}_2, \widehat{v}_3)$	$(\widehat{v}_3, \widehat{v}_4)$	$(\widehat{v}_3, \widehat{v}_5)$	$(\widehat{v}_4, \widehat{v}_5)$
σ_1	$(0.2, 0.3, 0.2, 0.2)$	$(0.2, 0.1, 0.2, 0.2)$	$(0.2, 0.1, 0.3, 0.2)$	$(0, 0, 1, 0)$	$(0, 0, 1, 0)$	$(0, 0, 1, 0)$
σ_2	$(0.2, 0.3, 0.2, 0.2)$	$(0.2, 0.3, 0.1, 0.2)$	$(0.2, 0.2, 0.1, 0.2)$	$(0.2, 0.3, 0.1, 0.2)$	$(0, 0, 1, 0)$	$(0.3, 0.1, 0.2, 0.3)$
σ_3	$(0, 0, 1, 0)$	$(0.1, 0.2, 0.1, 0.2)$	$(0.1, 0.2, 0.1, 0.3)$	$(0, 0, 1, 0)$	$(0, 0, 1, 0)$	$(0, 0, 1, 0)$
σ_4	$(0, 0, 1, 0)$	$(0, 0, 1, 0)$	$(0, 0, 1, 0)$	$(0.2, 0.1, 0.3, 0.2)$	$(0.1, 0.2, 0.2, 0.3)$	$(0.1, 0.3, 0.2, 0.3)$

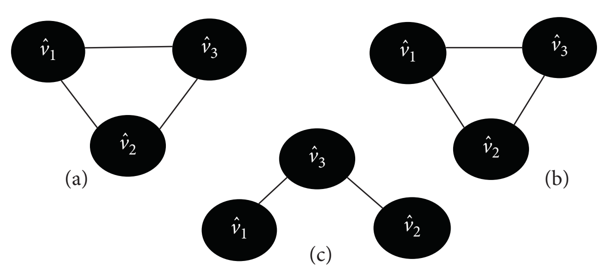


Figure 3: Graphical representation of Table 3.

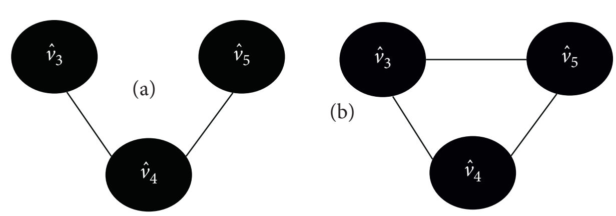


Figure 4: Geometrical depiction of Table 4.

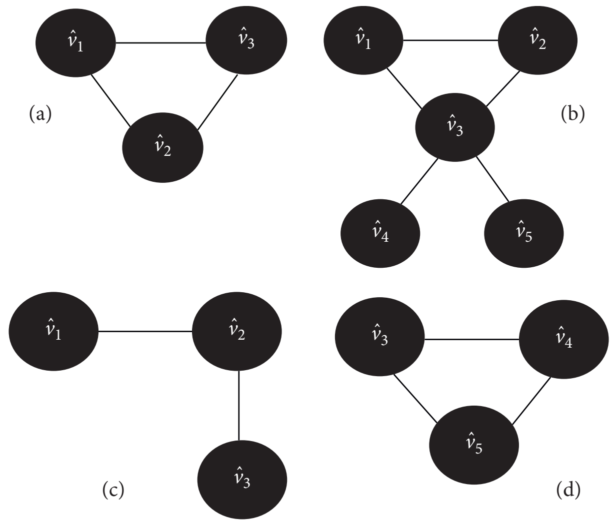


FIGURE 5: Graphical representation of Table 5.

$$
\begin{aligned}
\mathbb{T}_{\mathbb{J}_\sigma} &= \begin{cases} \mathbb{T}^1_{\mathbb{J}_\sigma}(\hat{v}) \text{if } \sigma \in \mathfrak{Q}^1 - \mathfrak{Q}^2, \\ \mathbb{T}^2_{\mathbb{J}_\sigma}(\hat{v}) \text{if } \sigma \in \mathfrak{Q}^2 - \mathfrak{Q}^1, \\ \min\left\{\mathbb{T}^1_{\mathbb{J}_\sigma}(\hat{v}), \mathbb{T}^2_{\mathbb{J}_\sigma}(\hat{v})\right\} \text{if } \sigma \in \mathfrak{Q}^1 \cap \mathfrak{Q}^2, \end{cases} \\
\mathbb{I}_{\mathbb{J}_\sigma} &= \begin{cases} \mathbb{I}^1_{\mathbb{J}_\sigma}(\hat{v}) \text{if } \sigma \in \mathfrak{Q}^1 - \mathfrak{Q}^2, \\ \mathbb{I}^2_{\mathbb{J}_\sigma}(\hat{v}) \text{if } \sigma \in \mathfrak{Q}^2 - \mathfrak{Q}^1, \\ \min\left\{\mathbb{I}^1_{\mathbb{J}_\sigma}(\hat{v}), \mathbb{I}^2_{\mathbb{J}_\sigma}(\hat{v})\right\} \text{if } \sigma \in \mathfrak{Q}^1 \cap \mathfrak{Q}^2, \end{cases} \\
\mathbb{F}_{\mathbb{J}_\sigma} &= \begin{cases} \mathbb{F}^1_{\mathbb{J}_\sigma}(\hat{v}) \text{if } \sigma \in \mathfrak{Q}^1 - \mathfrak{Q}^2, \\ \mathbb{F}^2_{\mathbb{J}_\sigma}(\hat{v}) \text{if } \sigma \in \mathfrak{Q}^2 - \mathfrak{Q}^1, \\ \max\left\{\mathbb{F}^1_{\mathbb{J}_\sigma}(\hat{v}), \mathbb{F}^2_{\mathbb{J}_\sigma}(\hat{v})\right\} \text{if } \sigma \in \mathfrak{Q}^1 \cap \mathfrak{Q}^2, \end{cases} \\
\mu_{\mathbb{J}_\sigma} &= \begin{cases} \mu^1_{\mathbb{J}_\sigma}(\hat{v}) \text{if } \sigma \in \mathfrak{Q}^1 - \mathfrak{Q}^2, \\ \mu^2_{\mathbb{J}_\sigma}(\hat{v}) \text{if } \sigma \in \mathfrak{Q}^2 - \mathfrak{Q}^1, \\ \min\left\{\mu^1_{\mathbb{J}_\sigma}(\hat{v}), \mu^2_{\mathbb{J}_\sigma}(\hat{v})\right\} \text{if } \sigma \in \mathfrak{Q}^1 \cap \mathfrak{Q}^2. \end{cases}
\end{aligned} \tag{3}
$$

The uncertain parts for $\mathbb{K}$ are given as follows:

$$
\begin{aligned}
\mathbb{T}_{\mathbb{K}_\sigma} &= \begin{cases} \mathbb{T}^1_{\mathbb{K}_\sigma}(\hat{v}) \text{if } \sigma \in \mathfrak{Q}^1 - \mathfrak{Q}^2, \\ \mathbb{T}^2_{\mathbb{K}_\sigma}(\hat{v}) \text{if } \sigma \in \mathfrak{Q}^2 - \mathfrak{Q}^1, \\ \min\left\{\mathbb{T}^1_{\mathbb{K}_\sigma}(\hat{v}), \mathbb{T}^2_{\mathbb{K}_\sigma}(\hat{v})\right\} \text{if } \sigma \in \mathfrak{Q}^1 \cap \mathfrak{Q}^2, \end{cases} \\
\mathbb{I}_{\mathbb{K}_\sigma} &= \begin{cases} \mathbb{I}^1_{\mathbb{K}_\sigma}(\hat{v}) \text{if } \sigma \in \mathfrak{Q}^1 - \mathfrak{Q}^2, \\ \mathbb{I}^2_{\mathbb{K}_\sigma}(\hat{v}) \text{if } \sigma \in \mathfrak{Q}^2 - \mathfrak{Q}^1, \\ \min\left\{\mathbb{I}^1_{\mathbb{K}_\sigma}(\hat{v}), \mathbb{I}^2_{\mathbb{K}_\sigma}(\hat{v})\right\} \text{if } \sigma \in \mathfrak{Q}^1 \cap \mathfrak{Q}^2, \end{cases} \\
\mathbb{F}_{\mathbb{K}_\sigma} &= \begin{cases} \mathbb{F}^1_{\mathbb{K}_\sigma}(\hat{v}) \text{if } \sigma \in \mathfrak{Q}^1 - \mathfrak{Q}^2, \\ \mathbb{F}^2_{\mathbb{K}_\sigma}(\hat{v}) \text{if } \sigma \in \mathfrak{Q}^2 - \mathfrak{Q}^1, \\ \max\left\{\mathbb{F}^1_{\mathbb{K}_\sigma}(\hat{v}), \mathbb{F}^2_{\mathbb{K}_\sigma}(\hat{v})\right\} \text{if } \sigma \in \mathfrak{Q}^1 \cap \mathfrak{Q}^2, \end{cases} \\
\mu_{\mathbb{K}_\sigma} &= \begin{cases} \mu^1_{\mathbb{K}_\sigma}(\hat{v}) \text{if } \sigma \in \mathfrak{Q}^1 - \mathfrak{Q}^2, \\ \mu^2_{\mathbb{K}_\sigma}(\hat{v}) \text{if } \sigma \in \mathfrak{Q}^2 - \mathfrak{Q}^1, \\ \min\left\{\mu^1_{\mathbb{K}_\sigma}(\hat{v}), \mu^2_{\mathbb{K}_\sigma}(\hat{v})\right\} \text{if } \sigma \in \mathfrak{Q}^1 \cap \mathfrak{Q}^2. \end{cases}
\end{aligned} \tag{4}
$$

Theorem 2. *If $\mathfrak{A}_1, \mathfrak{A}_2 \in \Omega_{PFHSG}$, then $\mathfrak{A}_1 \cap \mathfrak{A}_2 \in \Omega_{PFHSG}$.*

Proof. It can easily be proved with the help of axioms used in Definition 13. Therefore, its proof is omitted. □

Example 4. Let $\mathfrak{A}_1 = (\mathfrak{A}_1^\star, \mathfrak{Q}^1, \mathbb{J}^1, \mathbb{K}^1)$ be a popfhs-graph where $\mathfrak{A}_1^\star = (\mathfrak{B}_1, \mathfrak{E}_1)$ with $\mathfrak{B}_1 = \{\hat{v}_1, \hat{v}_2, \hat{v}_3\}$ and $\mathfrak{Q}_1, \mathfrak{Q}_2, \mathfrak{Q}_3$ are subparametric nonoverlapping sets with respect to distinct attributes $\alpha_1, \alpha_2, \alpha_3$, where $\mathfrak{Q}_1 = \{\alpha_{11}\}$, $\mathfrak{Q}_2 = \{\alpha_{21}\}$, and $\mathfrak{Q}_3 = \{\alpha_{31}, \alpha_{32}\}$. $\mathfrak{Q}^1 = \mathfrak{Q}_1 \times \mathfrak{Q}_2 \times \mathfrak{Q}_3 = \{\sigma_1, \sigma_2\}$ and $\mathbb{T}_{\mathbb{K}_\sigma}(\hat{v}_i, \hat{v}_j) = 0, \mathbb{I}_{\mathbb{K}_\sigma}(\hat{v}_i, \hat{v}_j) = 0, \mathbb{F}_{\mathbb{K}_\sigma}(\hat{v}_i, \hat{v}_j) = 1,$ and $(\forall (\hat{v}_i, \hat{v}_j) \in \mathfrak{B}_1 \times \mathfrak{B}_1 / \{(\hat{v}_1, \hat{v}_2), (\hat{v}_2, \hat{v}_3), (\hat{v}_1, \hat{v}_3)\})$. Table 6 and Figure 6 elaborate its tabulation and geometrical depiction, respectively. Also, let $\mathfrak{A}_2 = (\mathfrak{A}_2^\star, \mathfrak{Q}^2, \mathbb{J}^2, \mathbb{K}^2)$ be a popfhs-graph where $\mathfrak{A}_2^\star = (\mathfrak{B}_2, \mathfrak{E}_2)$ with $\mathfrak{B}_2 = \{\hat{v}_2, \hat{v}_3, \hat{v}_4\}$ and $\mathfrak{Q}_2, \mathfrak{Q}_3, \mathfrak{Q}_4$ are subparametric nonoverlapping sets with respect to distinct attributes $\alpha_2, \alpha_3, \alpha_4$ where $\mathfrak{Q}_2 = \{\alpha_{21}\}$, $\mathfrak{Q}_3 = \{\alpha_{31}, \alpha_{32}\}$, and $\mathfrak{Q}_4 = \{\alpha_{41}\}$. $\mathfrak{Q}^2 = \mathfrak{Q}_2 \times \mathfrak{Q}_3 \times \mathfrak{Q}_4 = \{\sigma_2, \sigma_3\}$ and $\mathcal{T}_{\mathbb{K}_\sigma}(\hat{v}_i, \hat{v}_j) = 0, \mathbb{I}_{\mathbb{K}_\sigma}(\hat{v}_i, \hat{v}_j) = 0, \mathbb{F}_{\mathbb{K}_\sigma}(\hat{v}_i, \hat{v}_j) = (1 \forall (\hat{v}_i, \hat{v}_j) \in \mathfrak{B}_2 \times \mathfrak{B}_2 / \{(\hat{v}_2, \hat{v}_3), (\hat{v}_3, \hat{v}_4), (\hat{v}_2, \hat{v}_4)\})$.

TABLE 6: Tabulation of popfhs-graph $\mathfrak{A}_1 = (\mathfrak{A}_1^\star, \mathfrak{Q}^1, \mathbb{J}^1, \mathbb{K}^1)$ for Example 4.

$\mathbb{J}$	$\hat{v}_1$	$\hat{v}_2$	$\hat{v}_3$
σ_1	$(0.2, 0.1, 0.2, 0.2)$	$(0.1, 0.2, 0.1, 0.3)$	$(0.1, 0.3, 0.2, 0.2)$
σ_2	$(0.1, 0.1, 0.3, 0.3)$	$(0.1, 0.2, 0.1, 0.5)$	$(0.2, 0.2, 0.4, 0.4)$
$\mathbb{K}$	$(\hat{v}_1, \hat{v}_2)$	$(\hat{v}_2, \hat{v}_3)$	$(\hat{v}_1, \hat{v}_3)$
σ_1	$(0.2, 0.2, 0.1, 0.2)$	$(0.2, 0.2, 0.1, 0.2)$	$(0.2, 0.1, 0.3, 0.2)$
σ_2	$(0.1, 0.3, 0.2, 0.3)$	$(0.2, 0.1, 0.3, 0.4)$	$(0.3, 0.1, 0.2, 0.3)$

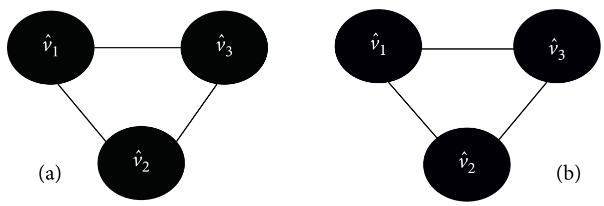


FIGURE 6: Geometrical interpretation of Table 6.

Its tabulation and pictorial representation are provided in Table 7 and Figure 7, respectively. Consider $\mathfrak{A} = \mathfrak{A}_1 \cap \mathfrak{A}_2$ with $\mathfrak{Q} = \mathfrak{Q}^1 \cap \mathfrak{Q}^2$. Its tabulation and geometrical interpretation are stated in Table 8 and Figure 8, respectively.

Definition 14. The compliment $\overline{\mathfrak{A}} = (\overline{\mathfrak{A}^\star}, \overline{\mathfrak{Q}}, \overline{\mathbb{F}}, \overline{\mathbb{G}})$ of SPFHS-subgraph $\mathfrak{A} = (\mathfrak{A}^\star, \mathfrak{Q}, \mathbb{J}, \mathbb{K})$ with $\mathbb{K}_\sigma(\hat{v}_1, \hat{v}_2) = \mathbb{J}_\sigma(\hat{v}_1) \cap \mathbb{J}_\sigma(\hat{v}_2)$ where

(1) $\overline{\mathfrak{Q}} = \mathfrak{Q}$

(2) $\overline{\mathbb{T}_{\mathbb{J}_\sigma}(\hat{v})} = \mathbb{T}_{\mathbb{J}_\sigma}(\hat{v}), \overline{\mathbb{I}_{\mathbb{J}_\sigma}(\hat{v})} = \mathbb{I}_{\mathbb{J}_\sigma}(\hat{v}), \overline{\mathbb{F}_{\mathbb{J}_\sigma}(\hat{v})} = \mathbb{F}_{\mathbb{J}_\sigma}(\hat{v})$, $\overline{\mu_{\mathbb{J}_\sigma}(\hat{v})} = \mu_{\mathbb{J}_\sigma}(\hat{v}) \forall \hat{v} \in \mathfrak{B}$

(3) $\overline{\mathbb{T}_{\mathbb{J}_\sigma}(\hat{v}_1, \hat{v}_2)} = \left\{\min\left\{\mathbb{T}_{\mathbb{J}_\sigma}(\hat{v}_1), \mathbb{T}_{\mathbb{J}_\sigma}(\hat{v}_2)\right\}, \text{if } \mathbb{T}_{\mathbb{K}_\sigma}(\hat{v}_1, \hat{v}_2) = 0,\ 0, \text{otherwise} \overline{\mathbb{I}_{\mathbb{J}_\sigma}(\hat{v}_1,\ \hat{v}_2)} = \left\{\min\left\{\mathbb{I}_{\mathbb{J}_\sigma}(\hat{v}_1), \mathbb{I}_{\mathbb{J}_\sigma}(\hat{v}_2)\right\}, \text{if } \mathbb{I}_{\mathbb{K}_\sigma}(\hat{v}_1, \hat{v}_2) = 0, 0, \text{otherwise}, \overline{\mathbb{F}_{\mathbb{J}_\sigma}(\hat{v}_1, \hat{v}_2)} = \left\{\max\left\{\mathbb{F}_{\mathbb{J}_\sigma}(\hat{v}_1), \mathbb{F}_{\mathbb{J}_\sigma}(\hat{v}_2)\right\}, \text{if } \mathbb{F}_{\mathbb{K}_\sigma}(\hat{v}_1, \hat{v}_2) = 0, 0, \text{otherwise}, \overline{\mu_{\mathbb{J}_\sigma}(\hat{v}_1, \hat{v}_2)} = \left\{\min\left\{\mu_{\mathbb{J}_\sigma}(\hat{v}_1), \mu_{\mathbb{J}_\sigma}(\hat{v}_2)\right\}, \text{if } \mu_{\mathbb{K}_\sigma}(\hat{v}_1, \hat{v}_2) = 0, 0, \text{otherwise}.$

4. Some Products and Composition of *popfhs*-Graphs

In this section, we discussed some products and composition of popfhs-graphs with the help of graphical representation and numerical examples.

Definition 15. For two *popfhs*-graphs, $\mathfrak{A}^1 = (\mathfrak{A}^{1\star}, \mathfrak{Q}^1, \mathbb{J}^1, \mathbb{K}^1)$ and $\mathfrak{A}^2 = (\mathfrak{A}^{2\star}, \mathfrak{Q}^2, \mathbb{J}^2, \mathbb{K}^2)$ with respect to $\mathfrak{A}^{1\star} = (\mathfrak{B}_1, \mathfrak{E}_1)$ and $\mathfrak{A}^{2\star} = (\mathfrak{B}_2, \mathfrak{E}_2)$. Let $\mathfrak{A} = \mathfrak{A}^1 \times_{\mathbb{P}} \mathfrak{A}^2$ where $\mathfrak{A} = (\mathbb{J}, \mathbb{K}, \mathfrak{Q}^1 \times \mathfrak{Q}^2)$ and $(\mathbb{J} = \mathbb{J}^1 \times \mathbb{J}^2, \mathbb{K} = \mathbb{K}^1 \times \mathbb{K}^2)$ is *popfhs*-set over $\mathfrak{B} = \mathfrak{B}_1 \times \mathfrak{B}_2$, $\mathbb{K} = (\mathbb{K}^1 \times \mathbb{K}^2, \mathfrak{Q}^1 \times \mathfrak{Q}^2)$ is *popfhs*-set over $\mathfrak{E} = \{((\hat{\pi}, \hat{\tau}_1), (\hat{\pi}, \hat{\tau}_2)) | \hat{\pi} \in \mathfrak{B}_1, (\hat{\tau}_1, \hat{\tau}_2) \in \mathfrak{E}_2\} \cup \{((\hat{\pi}_1, \hat{\tau}), (\hat{\pi}_2, \hat{\tau})) | \hat{\tau} \in \mathfrak{B}_2, (\hat{\pi}_1, \hat{\pi}_2) \in \mathfrak{E}_1\}$, and $\mathbb{K} = (\mathbb{J}, \mathbb{K}, \mathfrak{Q}^1 \times \mathfrak{Q}^2)$ is *popfhs*-graphs where

(1) $\mathbb{T}_{\mathbb{J}(\gamma,\zeta)}(\hat{\pi}, \hat{\tau}) = \mathbb{T}_{\mathbb{J}^1(\gamma)}(\hat{\pi}) \wedge \mathbb{T}_{\mathbb{J}^2(t)}(\hat{\tau}) \mathbb{I}_{\mathbb{J}(\gamma,\zeta)}(\hat{\pi}, \hat{\tau}) = \mathbb{I}_{\mathbb{J}^1(\gamma)}(\hat{\pi}) \wedge \mathbb{I}_{\mathbb{J}^2(t)}(\hat{\tau}) \mathbb{F}_{\mathbb{J}(\gamma,\zeta)}(\hat{\pi}, \hat{\tau}) = \mathbb{F}_{\mathbb{J}^1(\gamma)}(\hat{\pi}) \vee \mathbb{F}_{\mathbb{J}^2(t)}(\hat{\tau}) \mu_{\mathbb{J}(\gamma,\zeta)}(\hat{\pi}, \hat{\tau}) = \mu_{\mathbb{J}^1(\gamma)}(\hat{\pi}) \wedge \mu_{\mathbb{J}^2(t)}(\hat{\tau}) \forall (\hat{\pi}, \hat{\tau}) \in \mathfrak{B}$, $(s, t) \in \mathfrak{Q}^1 \times \mathfrak{Q}^2$

TABLE 7: Tabulation of popfhs-graph $\mathfrak{A}_2 = (\mathfrak{A}_2^\star, \mathfrak{Q}^2, \mathbb{J}^2, \mathbb{K}^2)$ for Example 4.

$\mathbb{J}$	$\hat{v}_2$	$\hat{v}_3$	$\hat{v}_4$
σ_2	$(0.1, 0.2, 0.1, 0.4)$	$(0.2, 0.1, 0.3, 0.5)$	$(0.3, 0.1, 0.1, 0.3)$
σ_3	$(0.3, 0.2, 0.1, 0.3)$	$(0.1, 0.1, 0.3, 0.2)$	$(0.2, 0.1, 0.2, 0.2)$
$\mathbb{K}$	$(\hat{v}_2, \hat{v}_3)$	$(\hat{v}_3, \hat{v}_4)$	$(\hat{v}_2, \hat{v}_4)$
σ_2	$(0.2, 0.2, 0.3, 0.2)$	$(0.2, 0.1, 0.1, 0.2)$	$(0.2, 0.2, 0.3, 0.2)$
σ_3	$(0.1, 0.1, 0.4, 0.3)$	$(0.2, 0.1, 0.4, 0.4)$	$(0.5, 0.2, 0.1, 0.3)$

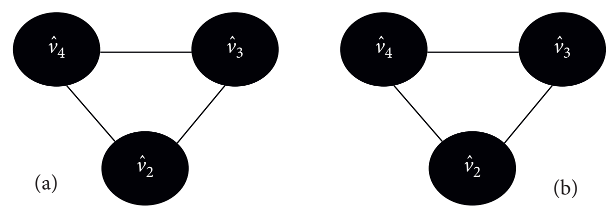


FIGURE 7: Graphical representation of Table 7.

TABLE 8: Tabulation of popfhs-graph $\mathfrak{A} = \mathfrak{A}_1 \cap \mathfrak{A}_2$.

$\mathbb{J}$	$\hat{v}_2$	$\hat{v}_3$
σ_1	$(0.1, 0.2, 0.1, 0.3)$	$(0.1, 0.3, 0.2, 0.2)$
σ_2	$(0.1, 0.2, 0.1, 0.4)$	$(0.2, 0.1, 0.4, 0.4)$
σ_3	$(0.3, 0.2, 0.1, 0.3)$	$(0.1, 0.1, 0.3, 0.2)$
$\mathbb{K}$	$(\hat{v}_2, \hat{v}_3)$	
σ_1	$(0.2, 0.2, 0.1, 0.2)$	
σ_2	$(0.2, 0.1, 0.3, 0.2)$	
σ_3	$(0.1, 0.1, 0.4, 0.3)$	

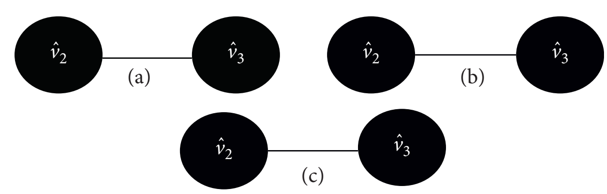


FIGURE 8: Graphical representation of Table 8.

(2) $\mathbb{T}_{\mathbb{K}(\gamma,\zeta)}((\hat{\pi}, \hat{\tau}_1), (\hat{\pi}, \hat{\tau}_2)) = \mathbb{T}_{\mathbb{J}^1(\gamma)}(\hat{\pi}) \wedge \mathbb{T}_{\mathbb{K}^2(\zeta)}(\hat{\tau}_1, \hat{\tau}_2)$ $\mathbb{I}_{\mathbb{K}(\gamma,\zeta)}((\hat{\pi}, \hat{\tau}_1), (\hat{\pi}, \hat{\tau}_2)) = \mathbb{I}_{\mathbb{J}^1(\gamma)}(\hat{\pi}) \wedge \mathbb{I}_{\mathbb{K}^2(\zeta)}(\hat{\tau}_1, \hat{\tau}_2)$ $\mathbb{F}_{\mathbb{K}(\gamma,\zeta)}((\hat{\pi}, \hat{\tau}_1), (\hat{\pi}, \hat{\tau}_2)) = \mathbb{F}_{\mathbb{J}^1(\gamma)}(\hat{\pi}) \vee \mathbb{F}_{\mathbb{K}^2(\zeta)}(\hat{\tau}_1, \hat{\tau}_2)$ $\mu_{\mathbb{K}(\gamma,\zeta)}((\hat{\pi}, \hat{\tau}_1), (\hat{\pi}, \hat{\tau}_2)) = \mu_{\mathbb{J}^1(\gamma)}(\hat{\pi}) \wedge \mu_{\mathbb{K}^2(\zeta)}(\hat{\tau}_1, \hat{\tau}_2) \forall \hat{\pi} \in \mathfrak{B}_1, (\hat{\tau}_1, \hat{\tau}_2) \in \mathfrak{E}_2$,

(3) $\mathbb{T}_{\mathbb{K}(\gamma,\zeta)}((\hat{\pi}_1, \hat{\tau}), (\hat{\pi}_2, \hat{\tau})) = \mathbb{T}_{\mathbb{J}^2(\zeta)}(\hat{\tau}) \wedge \mathbb{T}_{\mathbb{K}^1(\gamma)}(\hat{\pi}_1, \hat{\pi}_2) \mathbb{I}_{\mathbb{K}(\gamma,\zeta)}((\hat{\pi}_1, \hat{\tau}), (\hat{\pi}_2, \hat{\tau})) = \mathbb{I}_{\mathbb{J}^2(\zeta)}(\hat{\tau}) \wedge \mathbb{I}_{\mathbb{K}^1(\gamma)}(\hat{\pi}_1, \hat{\pi}_2) \mathbb{F}_{\mathbb{K}(\gamma,\zeta)}((\hat{\pi}_1, \hat{\tau}), (\hat{\pi}_2, \hat{\tau})) = \mathbb{F}_{\mathbb{J}^2(\zeta)}(\hat{\tau}) \wedge \mathbb{F}_{\mathbb{K}^1(\gamma)}(\hat{\pi}_1, \hat{\pi}_2) \mu_{\mathbb{K}(\gamma,\zeta)}((\hat{\pi}_1, \hat{\tau}), (\hat{\pi}_2, \hat{\tau})) = \mu_{\mathbb{J}^2(\zeta)}(\hat{\tau}) \wedge \mu_{\mathbb{K}^1(\gamma)}(\hat{\pi}_1, \hat{\pi}_2) \forall \hat{\tau} \in \mathfrak{B}_2, (\hat{\pi}_1, \hat{\pi}_2) \in \mathfrak{E}_1$,

$\forall (\gamma, \zeta) \in \mathfrak{Q}^1 \times \mathfrak{Q}^2$ and $\mathbb{W}(\gamma, \zeta) = \mathbb{W}_1(\gamma) \times \mathbb{W}_2(\zeta)$ are *popfhs*-graphs of $\mathfrak{A}$.

Example 5. Let $\mathfrak{A}^{1\star} = (\mathfrak{B}_1, \mathfrak{E}_1)$ be a simple graph with $\mathfrak{B}_1 = \{\hat{\pi}_1, \hat{\pi}_2, \hat{\pi}_3\}$ and $\mathfrak{E}_1 = \{\hat{\pi}_1\hat{\pi}_2, \hat{\pi}_1\hat{\pi}_3, \hat{\pi}_2\hat{\pi}_3\}$ and $\mathfrak{Q}_1, \mathfrak{Q}_2, \mathfrak{Q}_3$ are subparametric nonoverlapping sets with respect to distinct attributes $\alpha_1, \alpha_2, \alpha_3$ where $\mathfrak{Q}_1 = \{\alpha_{11}\}$, $\mathfrak{Q}_2 = \{\alpha_{21}, \alpha_{22}\}$, and $\mathfrak{Q}_3 = \{\alpha_{31}\}$. $\mathfrak{Q}^1 = \mathfrak{Q}_1 \times \mathfrak{Q}_2 \times \mathfrak{Q}_3 = \{\hat{\omega}_1, \hat{\omega}_2\}$. $\mathfrak{A}^1 = \{(\mathbb{W}_1,$

Table 9: popfhs-graph $\mathfrak{A}^1 = (\mathfrak{A}^\star, \mathfrak{Q}^1, \mathbb{J}^1, \mathbb{K}^1)$ (Example 5).

$\mathbb{J}$	$\hat{\pi}_1$	$\hat{\pi}_2$	$\hat{\pi}_3$
$\hat{\omega}_1$	(0.3, 0.2, 0.2, 0.3)	(0.2, 0.1, 0.3, 0.5)	(0.2, 0.3, 0.3, 0.5)
$\hat{\omega}_2$	(0.2, 0.1, 0.1, 0.4)	(0.2, 0.2, 0.3, 0.5)	(0.2, 0.3, 0.1, 0.6)
$\mathbb{K}$	$(\hat{\pi}_1, \hat{\pi}_2)$	$(\hat{\pi}_1, \hat{\pi}_3)$	$(\hat{\pi}_2, \hat{\pi}_3)$
$\hat{\omega}_1$	(0.2, 0.1, 0.4, 0.3)	(0.2, 0.3, 0.2, 0.2)	(0.2, 0.4, 0.1, 0.3)
$\hat{\omega}_2$	(0.3, 0.2, 0.2, 0.3)	(0.2, 0.4, 0.2, 0.3)	(0, 0, 1, 0)

$\mathfrak{Q}^1)\} = \{(\mathbb{W}_1(\hat{\omega}_1)), (\mathbb{W}_1(\hat{\omega}_2))\}$ is popfhs -graph, which is stated in Table 9.

Let $\mathfrak{A}^{2\star} = (\mathfrak{B}_2, \mathfrak{E}_2)$ be a simple graph with $\mathfrak{B}_2 = \{\hat{\tau}_1, \hat{\tau}_2, \hat{\tau}_3, \hat{\tau}_4\}$ and $\mathfrak{E}_2 = \{\hat{\tau}_1\hat{\tau}_2, \hat{\tau}_1\hat{\tau}_3, \hat{\tau}_1\hat{\tau}_4, \hat{\tau}_3\hat{\tau}_4\}$, and $\mathfrak{Q}_1, \mathfrak{Q}_2, \mathfrak{Q}_3$ are subparametric nonoverlapping sets with respect to distinct attributes $\alpha_1, \alpha_2, \alpha_3$, where $\mathfrak{Q}_1 = \{\alpha_{11}\}$, $\mathfrak{Q}_2 = \{\alpha_{21}, \alpha_{22}\}$, and $\mathfrak{Q}_3 = \{\alpha_{31}\}$. $\mathfrak{Q}^2 = \mathfrak{Q}_1 \times \mathfrak{Q}_2 \times \mathfrak{Q}_3 = \{\hat{\omega}_3, \hat{\omega}_4\}$. $\mathfrak{A}^2 = \{(\mathbb{W}_2, \mathfrak{Q}^2)\} = \{(\mathbb{W}_2(\hat{\omega}_3)), (\mathbb{W}_2(\hat{\omega}_4))\}$ is $popfhs$-graph that is depicted in Table 10.

$\mathfrak{A} = \mathfrak{A}^1 \times_{\mathbb{P}} \mathfrak{A}^2 = (\mathbb{W}, \mathfrak{Q}^1 \times \mathfrak{Q}^2)$, where $\mathfrak{Q}^1 \times \mathfrak{Q}^2 = \{(\hat{\omega}_1, \hat{\omega}_3), (\hat{\omega}_2, \hat{\omega}_3), (\hat{\omega}_1, \hat{\omega}_4), (\hat{\omega}_2, \hat{\omega}_4)\}$. Here, $\mathbb{W}(\hat{\omega}_1, \hat{\omega}_3) = \mathbb{W}_1(\hat{\omega}_1) \times_{\mathbb{P}} \mathbb{W}_2(\hat{\omega}_3)$, $\mathbb{W}(\hat{\omega}_2, \hat{\omega}_3) = \mathbb{W}_1(\hat{\omega}_2) \times_{\mathbb{P}} \mathbb{W}_2(\hat{\omega}_3)$, $\mathbb{W}(\hat{\omega}_1, \hat{\omega}_4) = \mathbb{W}_1(\hat{\omega}_1) \times_{\mathbb{P}} \mathbb{W}_2(\hat{\omega}_4)$, and $\mathbb{W}(\hat{\omega}_2, \hat{\omega}_4) = \mathbb{W}_1(\hat{\omega}_2) \times_{\mathbb{P}} \mathbb{W}_2(\hat{\omega}_4)$; for convenience we will write $(\hat{\pi}_p, \hat{\tau}_q) = \hat{\eta}_{pq}$ for $p = 1, 2, 3$ and $q = 1, 2, 3, 4$ also $\mathbb{T}_{\mathbb{K}_{\hat{\omega}}}(\hat{\eta}_i, \hat{\eta}_j) = 0, \mathbb{I}_{\mathbb{K}_{\hat{\omega}}}(\hat{\eta}_i, \hat{\eta}_j) = 0, \mathbb{F}_{\mathbb{K}_{\hat{\omega}}}(\hat{\eta}_i, \hat{\eta}_j) = 1 \forall (\hat{\eta}_{pq}) \in \mathfrak{B} \times \mathfrak{B}$.

$$\left\{\begin{array}{c} (\hat{\eta}_{11}, \hat{\eta}_{12}), (\hat{\eta}_{11}, \hat{\eta}_{13}), (\hat{\eta}_{11}, \hat{\eta}_{21}), (\hat{\eta}_{11}, \hat{\eta}_{31}), (\hat{\eta}_{12}, \hat{\eta}_{22}), \\ (\hat{\eta}_{12}, \hat{\eta}_{32}), (\hat{\eta}_{13}, \hat{\eta}_{23}), (\hat{\eta}_{13}, \hat{\eta}_{33}), (\hat{\eta}_{13}, \hat{\eta}_{14}), (\hat{\eta}_{14}, \hat{\eta}_{24}), \\ (\hat{\eta}_{14}, \hat{\eta}_{34}), (\hat{\eta}_{21}, \hat{\eta}_{22}), (\hat{\eta}_{21}, \hat{\eta}_{23}), (\hat{\eta}_{21}, \hat{\eta}_{31}), (\hat{\eta}_{22}, \hat{\eta}_{32}), \\ (\hat{\eta}_{23}, \hat{\eta}_{24}), (\hat{\eta}_{23}, \hat{\eta}_{33}), (\hat{\eta}_{24}, \hat{\eta}_{34}), (\hat{\eta}_{31}, \hat{\eta}_{32}), (\hat{\eta}_{31}, \hat{\eta}_{33}), \\ (\hat{\eta}_{33}, \hat{\eta}_{34}) \end{array}\right\}, \tag{5}$$

popfhs-graph of $\mathbb{W}(\hat{\omega}_1, \hat{\omega}_3) = \mathbb{W}_1(\hat{\omega}_1) \times_{\mathbb{P}} \mathbb{W}_2(\hat{\omega}_3)$ is given in Table 11.

The graphical representation of Table 9 is provided in Figure 9.

The graphical explanations of Tables 10 and 11 are presented in Figures 10 and 11, respectively.

Theorem 3. *The Cartesian product of two popfhs-graphs is popfhs-graph.*

Proof. It can easily be proved by following the axiomatic concepts provided in Definition 15 and Example 5. Therefore, we have omitted its proof. □

Definition 16. For two popfhs-graphs, $\mathfrak{A}^1 = (\mathfrak{A}^{1\star}, \mathfrak{Q}^1, \mathbb{J}^1, \mathbb{K}^1)$ and $\mathfrak{A}^2 = (\mathfrak{A}^{2\star}, \mathfrak{Q}^2, \mathbb{J}^2, \mathbb{K}^2)$ with respect to $\mathfrak{A}^{1\star} = (\mathfrak{B}_1, \mathfrak{E}_2)$ and $\mathfrak{A}^{2\star} = (\mathfrak{B}_2, \mathfrak{E}_2)$. Let $\mathfrak{A} = \mathfrak{A}^1 \circledcirc_{\mathbb{P}} \mathfrak{A}^2$ be cross product $\mathfrak{A}^1$ and $\mathfrak{A}^2$ where $\mathfrak{A} = (\mathbb{J}, \mathbb{K}, \mathfrak{Q}^1 \times \mathfrak{Q}^2)$ is popfhs-set over $\mathfrak{B} = \mathfrak{B}_1 \times \mathfrak{B}_2$, $\mathbb{K} = (\mathbb{K}^1 \circledcirc_{\mathbb{P}} \mathbb{K}^2, \mathfrak{Q}^1 \times \mathfrak{Q}^2)$ is popfhs-set over $\mathfrak{E} = \{((\hat{\pi}_1, \hat{\tau}_1), (\hat{\pi}_2, \hat{\tau}_2)) | (\hat{\pi}_1, \hat{\pi}_2) \in \mathfrak{E}_1, (\hat{\tau}_1, \hat{\tau}_2) \in \mathfrak{E}_2\}$, and $\mathbb{K} = (\mathbb{K}^1 \circledcirc_{\mathbb{P}} \mathbb{K}^2, \mathfrak{Q}^1 \times \mathfrak{Q}^2)$ is popfhs-graph where

(1) $\mathbb{T}_{\mathbb{J}(\gamma,\zeta)}(\hat{\pi}, \hat{\tau}) = \mathbb{T}_{\mathbb{J}^1(\gamma)}(\hat{\pi}) \wedge \mathbb{T}_{\mathbb{J}^2(t)}(\hat{\tau}) \mathbb{I}_{\mathbb{J}(\gamma,\zeta)}(\hat{\pi}, \hat{\tau}) = \mathbb{I}_{\mathbb{J}^1(\gamma)}(\hat{\pi}) \wedge \mathbb{I}_{\mathbb{J}^2(t)}(\hat{\tau}) \mathbb{F}_{\mathbb{J}(\gamma,\zeta)}(\hat{\pi}, \hat{\tau}) = \mathbb{F}_{\mathbb{J}^1(\gamma)}(\hat{\pi}) \vee \mathbb{F}_{\mathbb{J}^2(t)}(\hat{\tau}) \mu_{\mathbb{J}(\gamma,\zeta)}(\hat{\pi}, \hat{\tau}) = \mu_{\mathbb{J}^1(\gamma)}(\hat{\pi}) \wedge \mu_{\mathbb{J}^2(t)}(\hat{\tau}) \forall (\hat{\pi}, \hat{\tau}) \in \mathfrak{B}, (s, t) \in \mathfrak{Q}^1 \times \mathfrak{Q}^2$

(2) $\mathbb{T}_{\mathbb{K}(\gamma,\zeta)}((\hat{\pi}_1, \hat{\tau}_1), (\hat{\pi}_2, \hat{\tau}_2)) = \mathbb{T}_{\mathbb{K}^1(\gamma)}(\hat{\pi}_1, \hat{\pi}_2) \wedge \mathbb{T}_{\mathbb{K}^2(\zeta)}(\hat{\tau}_1, \hat{\tau}_2)$ $\mathbb{I}_{\mathbb{K}(\gamma,\zeta)}((\hat{\pi}_1, \hat{\tau}_1), (\hat{\pi}_2, \hat{\tau}_2)) = \mathbb{I}_{\mathbb{K}^1(\gamma)}(\hat{\pi}_1, \hat{\pi}_2) \wedge \mathbb{I}_{\mathbb{K}^2(\zeta)}(\hat{\tau}_1, \hat{\tau}_2) \mathbb{F}_{\mathbb{K}(\gamma,\zeta)}((\hat{\pi}_1, \hat{\tau}_1), (\hat{\pi}_2, \hat{\tau}_2)) = \mathbb{F}_{\mathbb{K}^1(\gamma)}(\hat{\pi}_1, \hat{\pi}_2) \vee \mathbb{F}_{\mathbb{K}^2(\zeta)}(\hat{\tau}_1, \hat{\tau}_2) \mu_{\mathbb{K}(\gamma,\zeta)}((\hat{\pi}_1, \hat{\tau}_1), (\hat{\pi}_2, \hat{\tau}_2)) = \mu_{\mathbb{K}^1(\gamma)}(\hat{\pi}_1, \hat{\pi}_2) \wedge \mu_{\mathbb{K}^2(\zeta)}(\hat{\tau}_1, \hat{\tau}_2) \forall \hat{\pi}_1, \hat{\pi}_2 \in \mathfrak{E}_1, (\hat{\tau}_1, \hat{\tau}_2) \in \mathfrak{E}_2$

$\forall (\hat{\pi}, \hat{\tau}) \in \mathfrak{Q}^1 \times \mathfrak{Q}^2$ and $\mathbb{W}(\gamma, \zeta) = \mathbb{W}_1(\gamma) \circledcirc_{\mathbb{P}} \mathbb{W}_2(\zeta)$ are popfhs-graphs of $\mathfrak{A}$.

Theorem 4. *The cross product of two popfhs-graphs is popfhs-graph.*

Proof. It can easily be proved by following the axiomatic concepts provided in Definition 16. Therefore, we have omitted its proof. □

Definition 17. For two popfhs-graphs, $\mathfrak{A}^1 = (\mathfrak{A}^{1\star}, \mathfrak{Q}^1, \mathbb{J}^1, \mathbb{K}^1)$ and $\mathfrak{A}^2 = (\mathfrak{A}^{2\star}, \mathfrak{Q}^2, \mathbb{J}^2, \mathbb{K}^2)$ with respect to $\mathfrak{A}^{1\star} = (\mathfrak{B}_1, \mathfrak{E}_2)$ and $\mathfrak{A}^{2\star} = (\mathfrak{B}_2, \mathfrak{E}_2)$. Let $\mathfrak{A} = \mathfrak{A}^1 \odot_{\mathbb{P}} \mathfrak{A}^2$ be lexicographic product $\mathfrak{A}^1$ and $\mathfrak{A}^2$ where $\mathfrak{A} = (\mathbb{J}, \mathbb{K}, \mathfrak{Q}^1 \times \mathfrak{Q}^2)$ is popfhs-set over $\mathfrak{B} = \mathfrak{B}_1 \times \mathfrak{B}_2$, $\mathbb{K} = (\mathbb{K}^1 \odot_{\mathbb{P}} \mathbb{K}^2, \mathfrak{Q}^1 \times \mathfrak{Q}^2)$ is popfhs-set over

$$\mathfrak{E} = \{((\hat{\pi}, \hat{\tau}_1), (\hat{\pi}, \hat{\tau}_2)) | \hat{\pi} \in \mathfrak{B}_1, (\hat{\tau}_1, \hat{\tau}_2) \in \mathfrak{E}_2\} \cup \{((\hat{\pi}_1, \hat{\tau}_1), (\hat{\pi}_2, \hat{\tau}_2)) | (\hat{\pi}_1, \hat{\pi}_2) \in \mathfrak{E}_1, (\hat{\tau}_1, \hat{\tau}_2) \in \mathfrak{E}_2\}, \tag{6}$$

and $\mathbb{K} = (\mathbb{K}^1 \odot_{\mathbb{P}} \mathbb{K}^2, \mathfrak{Q}^1 \times \mathfrak{Q}^2)$ is popfhs-graphs where

(1) $\mathbb{T}_{\mathbb{J}(\gamma,\zeta)}(\hat{\pi}, \hat{\tau}) = \mathbb{T}_{\mathbb{J}^1(\gamma)}(\hat{\pi}) \wedge \mathbb{T}_{\mathbb{J}^2(t)}(\hat{\tau}) \mathbb{I}_{\mathbb{J}(\gamma,\zeta)}(\hat{\pi}, \hat{\tau}) = \mathbb{I}_{\mathbb{J}^1(\gamma)}(\hat{\pi}) \wedge \mathbb{I}_{\mathbb{J}^2(t)}(\hat{\tau}) \mathbb{F}_{\mathbb{J}(\gamma,\zeta)}(\hat{\pi}, \hat{\tau}) = \mathbb{F}_{\mathbb{J}^1(\gamma)}(\hat{\pi}) \vee \mathbb{F}_{\mathbb{J}^2(t)}(\hat{\tau}) \mu_{\mathbb{J}(\gamma,\zeta)}(\hat{\pi}, \hat{\tau}) = \mu_{\mathbb{J}^1(\gamma)}(\hat{\pi}) \wedge \mu_{\mathbb{J}^2(t)}(\hat{\tau}) \forall (\hat{\pi}, \hat{\tau}) \in \mathfrak{B}, (s, t) \in \mathfrak{Q}^1 \times \mathfrak{Q}^2$

(2) $\mathbb{T}_{\mathbb{K}(\gamma,\zeta)}((\hat{\pi}, \hat{\tau}_1), (\hat{\pi}, \hat{\tau}_2)) = \mathbb{T}_{\mathbb{J}^1(\gamma)}(\hat{\pi}) \wedge \mathbb{T}_{\mathbb{K}^2(\zeta)}(\hat{\tau}_1, \hat{\tau}_2) \mathbb{I}_{\mathbb{K}(\gamma,\zeta)}((\hat{\pi}, \hat{\tau}_1), (\hat{\pi}, \hat{\tau}_2)) = \mathbb{I}_{\mathbb{J}^1(\gamma)}(\hat{\pi}) \wedge \mathbb{I}_{\mathbb{K}^2(\zeta)}(\hat{\tau}_1, \hat{\tau}_2)$ $\mathbb{F}_{\mathbb{K}(\gamma,\zeta)}((\hat{\pi}, \hat{\tau}_1), (\hat{\pi}, \hat{\tau}_2)) = \mathbb{F}_{\mathbb{J}^1(\gamma)}(\hat{\pi}) \vee \mathbb{F}_{\mathbb{K}^2(\zeta)}(\hat{\tau}_1, \hat{\tau}_2)$ $\mu_{\mathbb{K}(\gamma,\zeta)}((\hat{\pi}, \hat{\tau}_1), (\hat{\pi}, \hat{\tau}_2)) = \mu_{\mathbb{J}^1(\gamma)}(\hat{\pi}) \wedge \mu_{\mathbb{K}^2(\zeta)}(\hat{\tau}_1, \hat{\tau}_2) \forall \hat{\pi} \in \mathfrak{B}_1, (\hat{\tau}_1, \hat{\tau}_2) \in \mathfrak{E}_2$

(3) $\mathbb{T}_{\mathbb{K}(\gamma,\zeta)}((\hat{\pi}_1, \hat{\tau}_1), (\hat{\pi}_2, \hat{\tau}_2)) = \mathbb{T}_{\mathbb{K}^1(\gamma)}(\hat{\pi}_1, \hat{\pi}_2) \wedge \mathbb{T}_{\mathbb{K}^2(\zeta)}(\hat{\tau}_1, \hat{\tau}_2) \mathbb{I}_{\mathbb{K}(\gamma,\zeta)}((\hat{\pi}_1, \hat{\tau}_1), (\hat{\pi}_2, \hat{\tau}_2)) = \mathbb{I}_{\mathbb{K}^1(\gamma)}(\hat{\pi}_1, \hat{\pi}_2) \wedge \mathbb{I}_{\mathbb{K}^2(\zeta)}(\hat{\tau}_1, \hat{\tau}_2) \mathbb{F}_{\mathbb{K}(\gamma,\zeta)}((\hat{\pi}_1, \hat{\tau}_1), (\hat{\pi}_2, \hat{\tau}_2)) = \mathbb{F}_{\mathbb{K}^1(\gamma)}(\hat{\pi}_1, \hat{\pi}_2) \vee \mathbb{F}_{\mathbb{K}^2(\zeta)}(\hat{\tau}_1, \hat{\tau}_2) \mu_{\mathbb{K}(\gamma,\zeta)}((\hat{\pi}_1, \hat{\tau}_1), (\hat{\pi}_2, \hat{\tau}_2)) = \mu_{\mathbb{K}^1(\gamma)}(\hat{\pi}_1, \hat{\pi}_2) \wedge \mu_{\mathbb{K}^2(\zeta)}(\hat{\tau}_1, \hat{\tau}_2) \forall (\hat{\pi}_1, \hat{\pi}_2) \in \mathfrak{E}_1, (\hat{\tau}_1, \hat{\tau}_2) \in \mathfrak{E}_2$

Table 10: popfhs-graph $\mathfrak{A}^2 = (\mathfrak{A}^\star, \mathfrak{Q}^2, \mathbb{J}^2, \mathbb{K}^2)$ (Example 5).

$\mathbb{J}$	$\widehat{\tau}_1$	$\widehat{\tau}_2$	$\widehat{\tau}_3$	$\widehat{\tau}_4$		
$\widehat{\omega}_3$	$(0.3, 0.2, 0.2, 0.5)$	$(0.2, 0.1, 0.2, 0.4)$	$(0.3, 0.1, 0.1, 0.4)$	$(0.3, 0.1, 0.2, 0.6)$		
$\widehat{\omega}_4$	$(0.1, 0.1, 0.3, 0.5)$	$(0.3, 0.2, 0.2, 0.7)$	$(0.1, 0.2, 0.1, 0.5)$	$(0.2, 0.3, 0.1, 0.8)$		
$\mathbb{K}$	$(\widehat{\tau}_1, \widehat{\tau}_2)$	$(\widehat{\tau}_1, \widehat{\tau}_3)$	$(\widehat{\tau}_1, \widehat{\tau}_4)$	$(\widehat{\tau}_2, \widehat{\tau}_3)$	$(\widehat{\tau}_2, \widehat{\tau}_4)$	$(\widehat{\tau}_3, \widehat{\tau}_4)$
$\widehat{\omega}_3$	$(0.1, 0.2, 0.1, 0.3)$	$(0.2, 0.1, 0.2, 0.3)$	$(0, 0, 1, 0)$	$(0, 0, 1, 0)$	$(0, 0, 1, 0)$	$(0.1, 0.3, 0.1, 0.3)$
$\widehat{\omega}_4$	$(0.2, 0.1, 0.2, 0.4)$	$(0.3, 0.1, 0.1, 0.3)$	$(0.1, 0.2, 0.2, 0.4)$	$(0, 0, 1, 0)$	$(0, 0, 1, 0)$	$(0, 0, 1, 0)$

Table 11: popfhs-graph $\mathbb{W}(\widehat{\omega}_1, \widehat{\omega}_3) = \mathbb{W}_1(\widehat{\omega}_1) \times_{\mathbb{P}} \mathbb{W}_2(\widehat{\omega}_3)$ of $\mathfrak{A} = \mathfrak{A}^1 \times_{\mathbb{P}} \mathfrak{A}^2$ (Example 5).

$\mathbb{J}$	$\widehat{\eta}_{11}$	$\widehat{\eta}_{12}$	$\widehat{\eta}_{13}$	$\widehat{\eta}_{14}$	$\widehat{\eta}_{21}$	$\widehat{\eta}_{22}$
$(\widehat{\omega}_1, \widehat{\omega}_3)$	$(0.3, 0.2, 0.2, 0.3)$	$(0.2, 0.1, 0.2, 0.3)$	$(0.3, 0.1, 0.2, 0.3)$	$(0.3, 0.1, 0.2, 0.3)$	$(0.2, 0.1, 0.3, 0.5)$	$(0.2, 0.1, 0.3, 0.4)$
$\mathbb{J}$	$\widehat{\eta}_{23}$	$\widehat{\eta}_{24}$	$\widehat{\eta}_{31}$	$\widehat{\eta}_{32}$	$\widehat{\eta}_{33}$	$\widehat{\eta}_{34}$
$(\widehat{\omega}_1, \widehat{\omega}_3)$	$(0.2, 0.1, 0.3, 0.4)$	$(0.2, 0.1, 0.3, 0.5)$	$(0.2, 0.2, 0.3, 0.5)$	$(0.2, 0.1, 0.3, 0.4)$	$(0.2, 0.1, 0.3, 0.4)$	$(0.2, 0.1, 0.3, 0.5)$
$\mathbb{K}$	$(\widehat{\eta}_{11}, \widehat{\eta}_{12})$	$(\widehat{\eta}_{11}, \widehat{\eta}_{13})$	$(\widehat{\eta}_{11}, \widehat{\eta}_{21})$	$(\widehat{\eta}_{11}, \widehat{\eta}_{31})$	$(\widehat{\eta}_{12}, \widehat{\eta}_{22})$	$(\widehat{\eta}_{12}, \widehat{\eta}_{32})$
$(\widehat{\omega}_1, \widehat{\omega}_3)$	$(0.1, 0.2, 0.2, 0.3)$	$(0.2, 0.1, 0.2, 0.3)$	$(0.2, 0.1, 0.4, 0.3)$	$(0.2, 0.2, 0.2, 0.2)$	$(0.2, 0.1, 0.4, 0.3)$	$(0.2, 0.1, 0.2, 0.2)$
$\mathbb{K}$	$(\widehat{\eta}_{13}, \widehat{\eta}_{33})$	$(\widehat{\eta}_{13}, \widehat{\eta}_{14})$	$(\widehat{\eta}_{14}, \widehat{\eta}_{24})$	$(\widehat{\eta}_{14}, \widehat{\eta}_{34})$	$(\widehat{\eta}_{21}, \widehat{\eta}_{22})$	$(\widehat{\eta}_{21}, \widehat{\eta}_{23})$
$(\widehat{\omega}_1, \widehat{\omega}_3)$	$(0.2, 0.1, 0.2, 0.2)$	$(0.1, 0.2, 0.2, 0.3)$	$(0.2, 0.1, 0.4, 0.3)$	$(0.2, 0.1, 0.2, 0.2)$	$(0.1, 0.1, 0.3, 0.3)$	$(0.2, 0.1, 0.3, 0.3)$
$\mathbb{K}$	$(\widehat{\eta}_{22}, \widehat{\eta}_{32})$	$(\widehat{\eta}_{23}, \widehat{\eta}_{24})$	$(\widehat{\eta}_{23}, \widehat{\eta}_{33})$	$(\widehat{\eta}_{24}, \widehat{\eta}_{34})$	$(\widehat{\eta}_{31}, \widehat{\eta}_{32})$	$(\widehat{\eta}_{31}, \widehat{\eta}_{33})$
$(\widehat{\omega}_1, \widehat{\omega}_3)$	$(0.2, 0.1, 0.2, 0.3)$	$(0.1, 0.1, 0.3, 0.3)$	$(0.2, 0.1, 0.1, 0.3)$	$(0.2, 0.1, 0.2, 0.3)$	$(0.1, 0.2, 0.3, 0.3)$	$(0.2, 0.1, 0.3, 0.3)$
$\mathbb{K}$	$(\widehat{\eta}_{13}, \widehat{\eta}_{23})$		$(\widehat{\eta}_{21}, \widehat{\eta}_{31})$		$(\widehat{\eta}_{33}, \widehat{\eta}_{34})$	
$(\widehat{\omega}_1, \widehat{\omega}_3)$	$(0.2, 0.1, 0.4, 0.3)$		$(0.2, 0.2, 0.2, 0.3)$		$(0.1, 0.3, 0.3, 0.3)$	

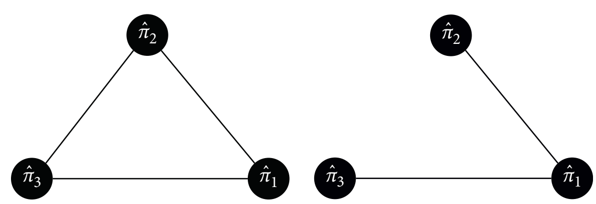
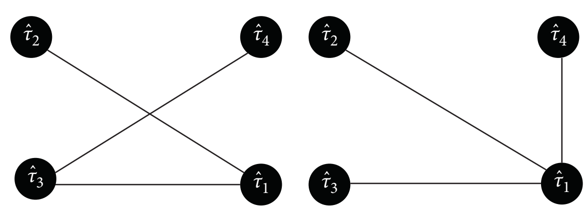


Figure 9: Graphical representation of Table 9 with (a) $\mathbb{W}(\widehat{\omega}_1)$ and (b) $\mathbb{W}(\widehat{\omega}_2)$.



(a) (b)

Figure 10: Geometrical depiction of Table 10 with (a) $\mathbb{W}(\widehat{\omega}_3)$ and (b) $\mathbb{W}(\widehat{\omega}_4)$.

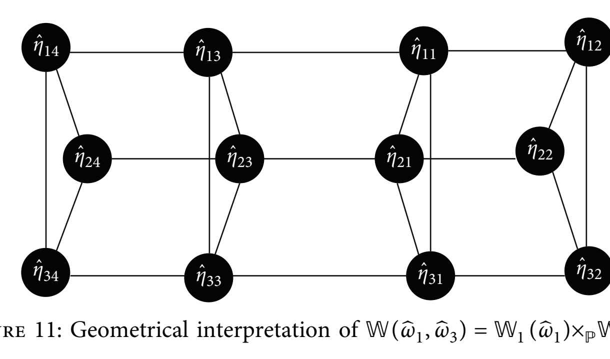


Figure 11: Geometrical interpretation of $\mathbb{W}(\widehat{\omega}_1, \widehat{\omega}_3) = \mathbb{W}_1(\widehat{\omega}_1) \times_{\mathbb{P}} \mathbb{W}_2(\widehat{\omega}_3)$.

Here, $\forall(\hat{\pi},\hat{\tau})\in\mathfrak{Q}^1\times\mathfrak{Q}^2$ and $\mathbb{W}(\gamma,\zeta)=\mathbb{W}_1(\gamma)\odot_{\mathbb{P}}\mathbb{W}_2(\zeta)$ are popfhs-graphs of $\mathfrak{A}$.

Theorem 5. *The lexicographical product of two popfhs-graphs is popfhs-graph.*

Proof. It can easily be proved by following the axiomatic concepts provided in Definition 17. Therefore, we have omitted its proof. □

Definition 18. For two popfhs-graphs, $\mathfrak{A}^1=(\mathfrak{A}^{1\star},\mathfrak{Q}^1,\mathbb{J}^1,\mathbb{K}^1)$ and $\mathfrak{A}^2=(\mathfrak{A}^{2\star},\mathfrak{Q}^2,\mathbb{J}^2,\mathbb{K}^2)$ with respect to $\mathfrak{A}^{1\star}=(\mathfrak{B}_1,\mathfrak{E}_2)$ and $\mathfrak{A}^{2\star}=(\mathfrak{B}_2,\mathfrak{E}_2)$. Let $\mathfrak{A}=\mathfrak{A}^1\otimes_{\mathbb{P}}\mathfrak{A}^2$ be strong product of $\mathfrak{A}^1$ and $\mathfrak{A}^2$, where $\mathfrak{A}=(\mathbb{J},\mathbb{K},\mathfrak{Q}^1\times\mathfrak{Q}^2)$ is popfhs-set over $\mathfrak{B}=\mathfrak{B}_1\times\mathfrak{B}_2$, $\mathbb{K}=(\mathbb{K}^1\otimes_{\mathbb{P}}\mathbb{K}^2,\mathfrak{Q}^1\times\mathfrak{Q}^2)$ is popfhs-set over

$$\begin{aligned}\mathfrak{E}=&\{((\hat{\pi},\hat{\tau}_1),(\hat{\pi},\hat{\tau}_2))|\hat{\pi}\in\mathfrak{B}_1,(\hat{\tau}_1,\hat{\tau}_2)\in\mathfrak{E}_2\}\cup\{((\hat{\pi}_1,\hat{\tau},)(\hat{\pi}_2,\hat{\tau}))|(\hat{\pi}_1,\hat{\pi}_2)\in\mathfrak{E}_1,\hat{\tau}\in\mathfrak{B}_2\}\\&\cup\{(\hat{\pi}_1,\hat{\tau}_1),(\hat{\pi}_2,\hat{\tau}_2)(\hat{\pi}_1,\hat{\pi}_2)\in\mathfrak{E}_1,(\hat{\tau}_1,\hat{\tau}_2)\in\mathfrak{E}_2\},\end{aligned}\tag{7}$$

and $\mathbb{K}=(\mathbb{K}^1\otimes_{\mathbb{P}}\mathbb{K}^2,\mathfrak{Q}^1\times\mathfrak{Q}^2)$ are popfhs-graphs where

(1) $\mathbb{T}_{\mathbb{J}(\gamma,\zeta)}(\hat{\pi},\hat{\tau})=\mathbb{T}_{\mathbb{J}^1(\gamma)}(\hat{\pi})\wedge\mathbb{T}_{\mathbb{J}^2(t)}(\hat{\tau})\mathbb{I}_{\mathbb{J}(\gamma,\zeta)}(\hat{\pi},\hat{\tau})=\mathbb{I}_{\mathbb{J}^1(\gamma)}(\hat{\pi})\wedge\mathbb{I}_{\mathbb{J}^2(t)}(\hat{\tau})\mathbb{F}_{\mathbb{J}(\gamma,\zeta)}(\hat{\pi},\hat{\tau})=\mathbb{F}_{\mathbb{J}^1(\gamma)}(\hat{\pi})\vee\mathbb{F}_{\mathbb{J}^2(t)}(\hat{\tau})\mu_{\mathbb{J}(\gamma,\zeta)}(\hat{\pi},\hat{\tau})=\mu_{\mathbb{J}^1(\gamma)}(\hat{\pi})\wedge\mu_{\mathbb{J}^2(t)}(\hat{\tau})\forall(\hat{\pi},\hat{\tau})\in\mathfrak{B},\ (s,t)\in\mathfrak{Q}^1\times\mathfrak{Q}^2$

(2) $\mathbb{T}_{\mathbb{K}(\gamma,\zeta)}((\hat{\pi},\hat{\tau}_1),(\hat{\pi},\hat{\tau}_2))=\mathbb{T}_{\mathbb{J}^1(\gamma)}(\hat{\pi})\wedge\mathbb{T}_{\mathbb{K}^2(\zeta)}(\hat{\tau}_1,\hat{\tau}_2)\mathbb{I}_{\mathbb{K}(\gamma,\zeta)}((\hat{\pi},\hat{\tau}_1),(\hat{\pi},\hat{\tau}_2))=\mathbb{I}_{\mathbb{J}^1(\gamma)}(\hat{\pi})\wedge\mathbb{I}_{\mathbb{K}^2(\zeta)}(\hat{\tau}_1,\hat{\tau}_2)\mathbb{F}_{\mathbb{K}(\gamma,\zeta)}((\hat{\pi},\hat{\tau}_1),(\hat{\pi},\hat{\tau}_2))=\mathbb{F}_{\mathbb{J}^1(\gamma)}(\hat{\pi})\vee\mathbb{F}_{\mathbb{K}^2(\zeta)}(\hat{\tau}_1,\hat{\tau}_2)\mu_{\mathbb{K}(\gamma,\zeta)}((\hat{\pi},\hat{\tau}_1),(\hat{\pi},\hat{\tau}_2))=\mu_{\mathbb{J}^1(\gamma)}(\hat{\pi})\wedge\mu_{\mathbb{K}^2(\zeta)}(\hat{\tau}_1,\hat{\tau}_2)\forall\hat{\pi}\in\ \mathfrak{B}_1,(\hat{\tau}_1,\hat{\tau}_2)\in\mathfrak{E}_2$

(3) $\mathbb{T}_{\mathbb{K}(\gamma,\zeta)}((\hat{\pi}_1,\hat{\tau}),(\hat{\pi}_2,\hat{\tau}))=\mathbb{T}_{\mathbb{K}^1(\gamma)}(\hat{\pi}_1,\hat{\pi}_2)\wedge\mathbb{T}_{\mathbb{J}^2(\zeta)}(\hat{\tau})\mathbb{I}_{\mathbb{K}(\gamma,\zeta)}((\hat{\pi}_1,\hat{\tau}),(\hat{\pi}_2,\hat{\tau}))=\mathbb{I}_{\mathbb{K}^1(\gamma)}(\hat{\pi}_1,\hat{\pi}_2)\wedge\mathbb{I}_{\mathbb{J}^2(\zeta)}(\hat{\tau})\mathbb{F}_{\mathbb{K}(\gamma,\zeta)}((\hat{\pi}_1,\hat{\tau}),(\hat{\pi}_2,\hat{\tau}))=\mathbb{F}_{\mathbb{K}^1(\gamma)}(\hat{\pi}_1,\hat{\pi}_2)\vee\mathbb{F}_{\mathbb{J}^2(\zeta)}(\hat{\tau})\mu_{\mathbb{K}(\gamma,\zeta)}((\hat{\pi}_1,\hat{\tau}),(\hat{\pi}_2,\hat{\tau}))=\mu_{\mathbb{K}^1(\gamma)}(\hat{\pi}_1,\hat{\pi}_2)\wedge\mu_{\mathbb{J}^2(\zeta)}(\hat{\tau})\forall\ (\hat{\pi}_1,\hat{\pi}_2)\in\mathfrak{E}_1,\hat{\tau}\in\mathfrak{B}_2$

(4) $\mathbb{T}_{\mathbb{K}(\gamma,\zeta)}((\hat{\pi}_1,\hat{\tau}_1),(\hat{\pi}_2,\hat{\tau}_2))=\mathbb{T}_{\mathbb{K}^1(\gamma)}(\hat{\pi}_1,\hat{\pi}_2)\wedge\mathbb{T}_{\mathbb{K}^2(\zeta)}(\hat{\tau}_1,\hat{\tau}_2)\mathbb{I}_{\mathbb{K}(\gamma,\zeta)}((\hat{\pi}_1,\hat{\tau}_1),(\hat{\pi}_2,\hat{\tau}_2))=\mathbb{I}_{\mathbb{K}^1(\gamma)}(\hat{\pi}_1,\hat{\pi}_2)\wedge\mathbb{I}_{\mathbb{K}^2(\zeta)}(\hat{\tau}_1,\hat{\tau}_2)\mathbb{F}_{\mathbb{K}(\gamma,\zeta)}((\hat{\pi}_1,\hat{\tau}_1),(\hat{\pi}_2,\hat{\tau}_2))=\mathbb{F}_{\mathbb{K}^1(\gamma)}(\hat{\pi}_1,\hat{\pi}_2)\vee\mathbb{F}_{\mathbb{K}^2(\zeta)}(\hat{\tau}_1,\hat{\tau}_2)\mu_{\mathbb{K}(\gamma,\zeta)}((\hat{\pi}_1,\hat{\tau}_1),(\hat{\pi}_2,\hat{\tau}_2))=\mu_{\mathbb{K}^1(\gamma)}(\hat{\pi}_1,\hat{\pi}_2)\wedge\mu_{\mathbb{K}^2(\zeta)}(\hat{\tau}_1,\hat{\tau}_2)\forall\qquad(\hat{\pi}_1,\hat{\pi}_2)\in\mathfrak{E}_1,(\hat{\tau}_1,\hat{\tau}_2)\in\mathfrak{E}_2$

Here, $\forall(\hat{\pi},\hat{\tau})\in\mathfrak{Q}^1\times\mathfrak{Q}^2$ and $\mathbb{W}(\gamma,\zeta)=\mathbb{W}_1(\gamma)\otimes_{\mathbb{P}}\mathbb{W}_2(\zeta)$ are popfhs-graphs of $\mathfrak{A}$.

Theorem 6. *The strong product of two popfhs-graphs is popfhs-graph.*

Definition 19. For two popfhs-graphs, $\mathfrak{A}^1=(\mathfrak{A}^{1\star},\mathfrak{Q}^1,\mathbb{J}^1,\mathbb{K}^1)$ and $\mathfrak{A}^2=(\mathfrak{A}^{2\star},\mathfrak{Q}^2,\mathbb{J}^2,\mathbb{K}^2)$ with respect to $\mathfrak{A}^{1\star}=(\mathfrak{B}_1,\mathfrak{E}_2)$ and $\mathfrak{A}^{2\star}=(\mathfrak{B}_2,\mathfrak{E}_2)$. Let $\mathfrak{A}=\mathfrak{A}^1[\mathfrak{A}^2]$ be composition of $\mathfrak{A}^1$ and $\mathfrak{A}^2$, where $\mathfrak{A}=(\mathbb{J},\mathbb{K},\mathfrak{Q}^1\times\mathfrak{Q}^2)$ is popfhs-set over $\mathfrak{B}=\mathfrak{B}_1\times\mathfrak{B}_2$, $\mathbb{K}=(\mathbb{K}^1\times\mathbb{K}^2,\mathfrak{Q}^1\times\mathfrak{Q}^2)$ is popfhs-set over

$$\begin{aligned}\mathfrak{E}=&\{((\hat{\pi},\hat{\tau}_1),(\hat{\pi},\hat{\tau}_2))|\hat{\pi}\in\mathfrak{B}_1,(\hat{\tau}_1,\hat{\tau}_2)\in\mathfrak{E}_2\}\cup\{(\hat{\pi}_1,\hat{\tau}),(\hat{\pi}_2,\hat{\tau})|(\hat{\pi}_1,\hat{\pi}_2)\in\mathfrak{E}_1,\hat{\tau}\in\mathfrak{B}_2\}\cup\\&\{((\hat{\pi}_1,\hat{\tau}_1),(\hat{\pi}_2,\hat{\tau}_2))|(\hat{\pi}_1,\hat{\pi}_2)\in\mathfrak{E}_1,\hat{\tau}_1\neq\hat{\tau}_2\},\end{aligned}\tag{8}$$

and $\mathbb{K}=(\mathbb{K}^1\times\mathbb{K}^2,\mathfrak{Q}^1\times\mathfrak{Q}^2)$ is popfhs-graphs where

(1) $\mathbb{T}_{\mathbb{J}(\gamma,\zeta)}(\hat{\pi},\hat{\tau})=\mathbb{T}_{\mathbb{J}^1(\gamma)}(\hat{\pi})\wedge\mathbb{T}_{\mathbb{J}^2(t)}(\hat{\tau})\mathbb{I}_{\mathbb{J}(\gamma,\zeta)}(\hat{\pi},\hat{\tau})=\mathbb{I}_{\mathbb{J}^1(\gamma)}(\hat{\pi})\wedge\mathbb{I}_{\mathbb{J}^2(t)}(\hat{\tau})\mathbb{F}_{\mathbb{J}(\gamma,\zeta)}(\hat{\pi},\hat{\tau})=\mathbb{F}_{\mathbb{J}^1(\gamma)}(\hat{\pi})\vee\mathbb{F}_{\mathbb{J}^2(t)}(\hat{\tau})\mu_{\mathbb{J}(\gamma,\zeta)}(\hat{\pi},\hat{\tau})=\mu_{\mathbb{J}^1(\gamma)}(\hat{\pi})\wedge\mu_{\mathbb{J}^2(t)}(\hat{\tau})\forall(\hat{\pi},\hat{\tau})\in\mathfrak{B},\ (s,t)\in\mathfrak{Q}^1\times\mathfrak{Q}^2$

(2) $\mathbb{T}_{\mathbb{K}(\gamma,\zeta)}((\hat{\pi},\hat{\tau}_1),(\hat{\pi},\hat{\tau}_2))=\quad\mathbb{T}_{\mathbb{J}^1(\gamma)}(\hat{\pi})\wedge\mathbb{T}_{\mathbb{K}^2(\zeta)}(\hat{\tau}_1,\hat{\tau}_2)\mathbb{I}_{\mathbb{K}(\gamma,\zeta)}((\hat{\pi},\hat{\tau}_1),(\hat{\pi},\hat{\tau}_2))=\mathbb{I}_{\mathbb{J}^1}(\gamma)(\hat{\pi})\wedge\mathbb{I}_{\mathbb{K}^2(\zeta)}(\hat{\tau}_1,\hat{\tau}_2)\mathbb{F}_{\mathbb{K}(\gamma,\zeta)}((\hat{\pi},\hat{\tau}_1),(\hat{\pi},\hat{\tau}_2))=\mathbb{F}_{\mathbb{J}^1(\gamma)}(\hat{\pi})\vee\mathbb{F}_{\mathbb{K}^2(\zeta)}(\hat{\tau}_1,\hat{\tau}_2)\mu_{\mathbb{K}(\gamma,\zeta)}((\hat{\pi},\hat{\tau}_1),(\hat{\pi},\hat{\tau}_2))=\mu_{\mathbb{J}^1(\gamma)}(\hat{\pi})\wedge\mu_{\mathbb{K}^2(\zeta)}(\hat{\tau}_1,\hat{\tau}_2)\forall\hat{\pi}\in\mathfrak{B}_1,(\hat{\tau}_1,\hat{\tau}_2)\in\ \mathfrak{E}_2$

(3) $\mathbb{T}_{\mathbb{K}(\gamma,\zeta)}((\hat{\pi}_1,\hat{\tau}),(\hat{\pi}_2,\hat{\tau}))=\mathbb{T}_{\mathbb{K}^1(\gamma)}(\hat{\pi}_1,\hat{\pi}_2)\wedge\mathbb{T}_{\mathbb{J}^2(\zeta)}(\hat{\tau})\mathbb{I}_{\mathbb{K}(\gamma,\zeta)}((\hat{\pi}_1,\hat{\tau}),(\hat{\pi}_2,\hat{\tau}))=\mathbb{I}_{\mathbb{K}^1(\gamma)}(\hat{\pi}_1,\hat{\pi}_2)\wedge\mathbb{I}_{\mathbb{J}^2(\zeta)}(\hat{\tau})\mathbb{F}_{\mathbb{K}(\gamma,\zeta)}((\hat{\pi}_1,\hat{\tau}),(\hat{\pi}_2,\hat{\tau}))=\mathbb{F}_{\mathbb{K}^1}(\gamma)(\hat{\pi}_1,\hat{\pi}_2)\vee\mathbb{F}_{\mathbb{J}^2(\zeta)}(\hat{\tau})\mu_{\mathbb{K}(\gamma,\zeta)}((\hat{\pi}_1,\hat{\tau}),(\hat{\pi}_2,\hat{\tau}))=\mu_{\mathbb{K}^1(\gamma)}(\hat{\pi}_1,\hat{\pi}_2)\wedge\mu_{\mathbb{J}^2(\zeta)}(\hat{\tau})\forall(\hat{\pi}_1,\hat{\pi}_2)\in\mathfrak{E}_1,\ \hat{\tau}\in\mathfrak{B}_2$

(4) $\mathbb{T}_{\mathbb{K}(\gamma,\zeta)}((\hat{\pi}_1,\hat{\tau}_1),(\hat{\pi}_2,\hat{\tau}_2))=\mathbb{T}_{\mathbb{K}^1(\gamma)}(\hat{\pi}_1,\hat{\pi}_2)\wedge\mathbb{T}_{\mathbb{J}^1(\zeta)}(\hat{\tau}_2)\wedge\mathbb{T}_{\mathbb{J}^2(\zeta)}(\hat{\tau}_1)\mathbb{I}_{\mathbb{K}(\gamma,\zeta)}((\hat{\pi}_1,\hat{\tau}_1),(\hat{\pi}_2,\hat{\tau}_2))=\mathbb{I}_{\mathbb{K}^1(\gamma)}(\hat{\pi}_1,\hat{\pi}_2)\wedge\mathbb{I}_{\mathbb{J}^1(\zeta)}(\hat{\tau}_2)\wedge\mathbb{I}_{\mathbb{J}^2(\zeta)}(\hat{\tau}_1)\mathbb{F}_{\mathbb{K}(\gamma,\zeta)}((\hat{\pi}_1,\hat{\tau}_1),(\hat{\pi}_2,\hat{\tau}_2))=\mathbb{F}_{\mathbb{K}^1(\gamma)}(\hat{\pi}_1,\hat{\pi}_2)\wedge\mathbb{F}_{\mathbb{J}^1(\zeta)}(\hat{\tau}_2)\wedge\mathbb{F}_{\mathbb{J}^2(\zeta)}(\hat{\tau}_1)\mu_{\mathbb{K}(\gamma,\zeta)}$

$$((\widehat{\pi}_1,\widehat{\tau}_1),(\widehat{\pi}_2,\widehat{\tau}_2)) = \mu_{\mathbb{K}^1(\gamma)} (\widehat{\pi}_1,\widehat{\pi}_2)\wedge\mu_{\mathbb{J}^1(\zeta)}(\widehat{\tau}_2)\wedge\mu_{\mathbb{J}^2(\zeta)}(\widehat{\tau}_1)\forall(\widehat{\pi}_1,\widehat{\pi}_2)\in\mathfrak{E}_1\widehat{\tau}_1\neq\widehat{\tau}_2$$

Here, $\forall(\gamma,\zeta)\in\mathfrak{Q}^1\times\mathfrak{Q}^2$ and $\mathbb{W}(\gamma,\zeta)=\mathbb{W}_1(\gamma)[\mathbb{W}_2(\zeta)]$ are popfhs-graphs of $\mathfrak{A}$.

Theorem 7. *The composition of two popfhs-graphs is popfhs-graph.*

Proof. It can easily be proved by following the axiomatic concepts provided in Definition 19. Therefore, we have omitted its proof. □

Definition 20. For two popfhs-graphs, $\mathfrak{A}^1=(\mathfrak{A}^{1\star},\mathfrak{Q}^1,\mathbb{J}^1,\mathbb{K}^1)$ and $\mathfrak{A}^2=(\mathfrak{A}^{2\star},\mathfrak{Q}^2,\mathbb{J}^2,\mathbb{K}^2)$ with respect to $\mathfrak{A}^{1\star}=(\mathfrak{B}_1,\mathfrak{E}_2)$ and $\mathfrak{A}^{2\star}=(\mathfrak{B}_2,\mathfrak{E}_2)$. Let $\mathfrak{A}=\mathfrak{A}^1\cup\mathfrak{A}^2$ be the union of $\mathfrak{A}^1$ and $\mathfrak{A}^2$ where $\mathfrak{A}=(\mathbb{J},\mathbb{K},\mathfrak{Q}^1\cup\mathfrak{Q}^2)$ is popfhs-set over $\mathfrak{B}=\mathfrak{B}_1\cup\mathfrak{B}_2$ and $\mathbb{K}=(\mathbb{K}^1\cup\mathbb{K}^2,\mathfrak{Q}^1\cup\mathfrak{Q}^2)$ is popfhs-set over $\mathfrak{E}=\mathfrak{E}_1\cup\mathfrak{E}_2$, where for $\widehat{\pi},\widehat{\tau}\in\mathfrak{B}$, $popf$-components are stated as follows:

$$\begin{gathered}
\mathbb{T}_{\mathbb{J}_{\widehat{\omega}}}(\widehat{\tau}) = \begin{cases} \mathbb{T}_{\mathbb{J}^1(\widehat{\omega})}(\widehat{\tau}) & ;\widehat{\omega}\in\mathfrak{Q}^1-\mathfrak{Q}^2 \\ \mathbb{T}_{\mathbb{J}^2(\widehat{\omega})}(\widehat{\tau}) & ;\widehat{\omega}\in\mathfrak{Q}^2-\mathfrak{Q}^1 \\ \mathbb{T}_{\mathbb{J}^1(\widehat{\omega})}(\widehat{\tau})\vee\mathbb{T}_{\mathbb{J}^2(\widehat{\omega})}(\widehat{\tau}) & ;\widehat{\omega}\in\mathfrak{Q}^1\cap\mathfrak{Q}^2 \end{cases} \\
\mathbb{I}_{\mathbb{J}_{\widehat{\omega}}}(\widehat{\tau}) = \begin{cases} \mathbb{I}_{\mathbb{J}^1(\widehat{\omega})}(\widehat{\tau}) & ;\widehat{\omega}\in\mathfrak{Q}^1-\mathfrak{Q}^2 \\ \mathbb{I}_{\mathbb{J}^2(\widehat{\omega})}(\widehat{\tau}) & ;\widehat{\omega}\in\mathfrak{Q}^2-\mathfrak{Q}^1 \\ \mathbb{I}_{\mathbb{J}^1(\widehat{\omega})}(\widehat{\tau})\vee\mathbb{I}_{\mathbb{J}^2(\widehat{\omega})}(\widehat{\tau}) & ;\widehat{\omega}\in\mathfrak{Q}^1\cap\mathfrak{Q}^2 \end{cases} \\
\mathbb{F}_{\mathbb{J}_{\widehat{\omega}}}(\widehat{\tau}) = \begin{cases} \mathbb{F}_{\mathbb{J}^1(\widehat{\omega})}(\widehat{\tau}) & ;\widehat{\omega}\in\mathfrak{Q}^1-\mathfrak{Q}^2 \\ \mathbb{F}_{\mathbb{J}^2(\widehat{\omega})}(\widehat{\tau}) & ;\widehat{\omega}\in\mathfrak{Q}^2-\mathfrak{Q}^1 \\ \mathbb{F}_{\mathbb{J}^1(\widehat{\omega})}(\widehat{\tau})\wedge\mathbb{F}_{\mathbb{J}^2(\widehat{\omega})}(\widehat{\tau}) & ;\widehat{\omega}\in\mathfrak{Q}^1\cap\mathfrak{Q}^2 \end{cases} \\
\mu_{\mathbb{J}_{\widehat{\omega}}}(\widehat{\tau}) = \begin{cases} \mu_{\mathbb{J}^1(\widehat{\omega})}(\widehat{\tau}) & ;\widehat{\omega}\in\mathfrak{Q}^1-\mathfrak{Q}^2 \\ \mu_{\mathbb{J}^2(\widehat{\omega})}(\widehat{\tau}) & ;\widehat{\omega}\in\mathfrak{Q}^2-\mathfrak{Q}^1 \\ \mu_{\mathbb{J}^1(\widehat{\omega})}(\widehat{\tau})\vee\mu_{\mathbb{J}^2(\widehat{\omega})}(\widehat{\tau}) & ;\widehat{\omega}\in\mathfrak{Q}^1\cap\mathfrak{Q}^2 \end{cases}
\end{gathered} \tag{1}$$

$$\begin{gathered}
\mathbb{T}_{\mathbb{K}_{\widehat{\omega}}}(\widehat{\pi}\widehat{\tau}) = \begin{cases} \mathbb{T}_{\mathbb{K}^1(\widehat{\omega})}(\widehat{\pi}\widehat{\tau}); & \widehat{\omega}\in\mathfrak{Q}^1-\mathfrak{Q}^2 \\ \mathbb{T}_{\mathbb{K}^2(\widehat{\omega})}(\widehat{\pi}\widehat{\tau}); & \widehat{\omega}\in\mathfrak{Q}^2-\mathfrak{Q}^1, \\ \mathbb{T}_{\mathbb{K}^1(\widehat{\omega})}(\widehat{\pi}\widehat{\tau})\vee\mathbb{T}_{\mathbb{K}^2(\widehat{\omega})}(\widehat{\pi}\widehat{\tau}); & \widehat{\omega}\in\mathfrak{Q}^1\cap\mathfrak{Q}^2, \end{cases} \\
\mathbb{I}_{\mathbb{K}_{\widehat{\omega}}}(\widehat{\pi}\widehat{\tau}) = \begin{cases} \mathbb{I}_{\mathbb{K}^1(\widehat{\omega})}(\widehat{\pi}\widehat{\tau}); & \widehat{\omega}\in\mathfrak{Q}^1-\mathfrak{Q}^2, \\ \mathbb{I}_{\mathbb{K}^2(\widehat{\omega})}(\widehat{\pi}\widehat{\tau}); & \widehat{\omega}\in\mathfrak{Q}^2-\mathfrak{Q}^1, \\ \mathbb{I}_{\mathbb{K}^1(\widehat{\omega})}(\widehat{\pi}\widehat{\tau})\vee\mathbb{I}_{\mathbb{K}^2(\widehat{\omega})}(\widehat{\pi}\widehat{\tau}); & \widehat{\omega}\in\mathfrak{Q}^1\cap\mathfrak{Q}^2, \end{cases} \\
\mathbb{F}_{\mathbb{K}_{\widehat{\omega}}}(\widehat{\pi}\widehat{\tau}) = \begin{cases} \mathbb{F}_{\mathbb{K}^1(\widehat{\omega})}(\widehat{\pi}\widehat{\tau}); & \widehat{\omega}\in\mathfrak{Q}^1-\mathfrak{Q}^2, \\ \mathbb{F}_{\mathbb{K}^2(\widehat{\omega})}(\widehat{\pi}\widehat{\tau}); & \widehat{\omega}\in\mathfrak{Q}^2-\mathfrak{Q}^1, \\ \mathbb{F}_{\mathbb{K}^1(\widehat{\omega})}(\widehat{\pi}\widehat{\tau})\vee\mathbb{I}_{\mathbb{K}^2(\widehat{\omega})}(\widehat{\pi}\widehat{\tau}); & \widehat{\omega}\in\mathfrak{Q}^1\cap\mathfrak{Q}^2, \end{cases} \\
\mu_{\mathbb{K}_{\widehat{\omega}}}(\widehat{\pi}\widehat{\tau}) = \begin{cases} \mu_{\mathbb{K}^1(\widehat{\omega})}(\widehat{\pi}\widehat{\tau}); & \widehat{\omega}\in\mathfrak{Q}^1-\mathfrak{Q}^2, \\ \mu_{\mathbb{K}^2(\widehat{\omega})}(\widehat{\pi}\widehat{\tau}); & \widehat{\omega}\in\mathfrak{Q}^2-\mathfrak{Q}^1, \\ \mu_{\mathbb{K}^1(\widehat{\omega})}(\widehat{\pi}\widehat{\tau})\vee\mathbb{I}_{\mathbb{K}^2(\widehat{\omega})}(\widehat{\pi}\widehat{\tau}); & \widehat{\omega}\in\mathfrak{Q}^1\cap\mathfrak{Q}^2. \end{cases}
\end{gathered} \tag{2}$$

Definition 21. For two popfhs-graphs, $\mathfrak{A}^1=(\mathfrak{A}^{1\star},\mathfrak{Q}^1,\mathbb{J}^1,\mathbb{K}^1)$ and $\mathfrak{A}^2=(\mathfrak{A}^{2\star},\mathfrak{Q}^2,\mathbb{J}^2,\mathbb{K}^2)$ with respect to $\mathfrak{A}^{1\star}=(\mathfrak{B}_1,\mathfrak{E}_2)$ and $\mathfrak{A}^{2\star}=(\mathfrak{B}_2,\mathfrak{E}_2)$. Let $\mathfrak{A}=\mathfrak{A}^1\cap\mathfrak{A}^2$ be the intersection of $\mathfrak{A}^1$ and $\mathfrak{A}^2$, where $\mathfrak{A}=(\mathbb{J},\mathbb{K},\mathfrak{Q}^1\cup\mathfrak{Q}^2)$ is popfhs-set over $\mathfrak{B}=\mathfrak{B}_1\cap\mathfrak{B}_2$ and $\mathbb{K}=(\mathbb{K}^1\cup\mathbb{K}^2,\mathfrak{Q}^1\cup\mathfrak{Q}^2)$ is popfhs-set over $\mathfrak{E}=\mathfrak{E}_1\cap\mathfrak{E}_2$, where for $\widehat{\pi},\widehat{\tau}\in\mathfrak{B}$, $popf$-components can be given by

$$\begin{gathered}
\varepsilon\mathbb{T}_{\mathbb{J}(\widehat{\omega})}(\widehat{\tau}) = \begin{cases} \mathbb{T}_{\mathbb{J}^1(\widehat{\omega})}(\widehat{\tau}) & ;\widehat{\omega}\in\mathfrak{Q}^1-\mathfrak{Q}^2 \\ \mathbb{T}_{\mathbb{J}(\widehat{\omega})}(\widehat{\tau}) & ;\widehat{\omega}\in\mathfrak{Q}^2-\mathfrak{Q}^1 \\ \mathbb{T}_{\mathbb{J}^1(\widehat{\omega})}(\widehat{\tau})\wedge\mathbb{T}_{\mathbb{J}^2(\widehat{\omega})}(\widehat{\tau}) & ;\widehat{\omega}\in\mathfrak{Q}^1\cap\mathfrak{Q}^2 \end{cases} \\
\mathbb{I}_{\mathbb{J}(\widehat{\omega})}(\widehat{\tau}) = \begin{cases} \mathbb{I}_{\mathbb{J}^1(\widehat{\omega})}(\widehat{\tau}) & ;\widehat{\omega}\in\mathfrak{Q}^1-\mathfrak{Q}^2 \\ \mathbb{I}_{\mathbb{J}^2(\widehat{\omega})}(\widehat{\tau}) & ;\widehat{\omega}\in\mathfrak{Q}^2-\mathfrak{Q}^1 \\ \mathbb{I}_{\mathbb{J}^1(\widehat{\omega})}(\widehat{\tau})\wedge\mathbb{I}_{\mathbb{J}(\widehat{\omega})}(\widehat{\tau}) & ;\widehat{\omega}\in\mathfrak{Q}^1\cap\mathfrak{Q}^2 \end{cases} \\
\mathbb{F}_{\mathbb{J}(\widehat{\omega})}(\widehat{\tau}) = \begin{cases} \mathbb{F}_{\mathbb{J}^1(\widehat{\omega})}(\widehat{\tau}) & ;\widehat{\omega}\in\mathfrak{Q}^1-\mathfrak{Q}^2 \\ \mathbb{F}_{\mathbb{J}(\widehat{\omega})}(\widehat{\tau}) & ;\widehat{\omega}\in\mathfrak{Q}^2-\mathfrak{Q}^1 \\ \mathbb{F}_{\mathbb{J}^1(\widehat{\omega})}(\widehat{\pi}\widehat{\tau})\vee\mathbb{F}_{\mathbb{J}^1(\widehat{\omega})}(\widehat{\pi}\widehat{\tau}) & ;\widehat{\omega}\in\mathfrak{Q}^1\cap\mathfrak{Q}^2 \end{cases} \\
\mu_{\mathbb{J}(\widehat{\omega})}(\widehat{\tau}) = \begin{cases} \mu_{\mathbb{J}^1(\widehat{\omega})}(\widehat{\tau}) & ;\widehat{\omega}\in\mathfrak{Q}^1-\mathfrak{Q}^2 \\ \mu_{\mathbb{J}(\widehat{\omega})}(\widehat{\tau}) & ;\widehat{\omega}\in\mathfrak{Q}^2-\mathfrak{Q}^1 \\ \mu_{\mathbb{J}^1(\widehat{\omega})}(\widehat{\tau})\wedge\mu_{\mathbb{J}^2(\widehat{\omega})}(\widehat{\tau}) & ;\widehat{\omega}\in\mathfrak{Q}^1\cap\mathfrak{Q}^2 \end{cases}
\end{gathered} \tag{1}$$

$$\begin{gathered}
\mathbb{T}_{\mathbb{K}_{\widehat{\omega}}}(\widehat{\pi}\widehat{\tau}) = \begin{cases} \mathbb{T}_{\mathbb{K}^1(\widehat{\omega})}(\widehat{\pi}\widehat{\tau}) & ;\widehat{\omega}\in\mathfrak{Q}^1-\mathfrak{Q}^2 \\ \mathbb{T}_{\mathbb{K}^2(\widehat{\omega})}(\widehat{\pi}\widehat{\tau}) & ;\widehat{\omega}\in\mathfrak{Q}^2-\mathfrak{Q}^1 \\ \mathbb{T}_{\mathbb{K}^1(\widehat{\omega})}(\widehat{\pi}\widehat{\tau})\wedge\mathbb{T}_{\mathbb{K}^2(\widehat{\omega})}(\widehat{\pi}\widehat{\tau}) & ;\widehat{\omega}\in\mathfrak{Q}^1\cap\mathfrak{Q}^2 \end{cases} \\
\mathbb{I}_{\mathbb{K}_{\widehat{\omega}}}(\widehat{\pi}\widehat{\tau}) = \begin{cases} \mathbb{I}_{\mathbb{K}^1(\widehat{\omega})}(\widehat{\pi}\widehat{\tau}) & ;\widehat{\omega}\in\mathfrak{Q}^1-\mathfrak{Q}^2 \\ \mathbb{I}_{\mathbb{K}^2(\widehat{\omega})}(\widehat{\pi}\widehat{\tau}) & ;\widehat{\omega}\in\mathfrak{Q}^2-\mathfrak{Q}^1 \\ \mathbb{I}_{\mathbb{K}^1(\widehat{\omega})}(\widehat{\pi}\widehat{\tau})\wedge\mathbb{I}_{\mathbb{K}^2(\widehat{\omega})}(\widehat{\pi}\widehat{\tau}) & ;\widehat{\omega}\in\mathfrak{Q}^1\cap\mathfrak{Q}^2 \end{cases} \\
\mathbb{F}_{\mathbb{K}_{\widehat{\omega}}}(\widehat{\pi}\widehat{\tau}) = \begin{cases} \mathbb{F}_{\mathbb{K}^1(\widehat{\omega})}(\widehat{\pi}\widehat{\tau}) & ;\widehat{\omega}\in\mathfrak{Q}^1-\mathfrak{Q}^2 \\ \mathbb{F}_{\mathbb{K}^2(\widehat{\omega})}(\widehat{\pi}\widehat{\tau}) & ;\widehat{\omega}\in\mathfrak{Q}^2-\mathfrak{Q}^1 \\ \mathbb{F}_{\mathbb{K}^1(\widehat{\omega})}(\widehat{\pi}\widehat{\tau})\vee\mathbb{F}_{\mathbb{K}^2(\widehat{\omega})}(\widehat{\pi}\widehat{\tau}) & ;\widehat{\omega}\in\mathfrak{Q}^1\cap\mathfrak{Q}^2 \end{cases} \\
\mu_{\mathbb{K}_{\widehat{\omega}}}(\widehat{\pi}\widehat{\tau}) = \begin{cases} \mu_{\mathbb{K}^1(\widehat{\omega})}(\widehat{\pi}\widehat{\tau}) & ;\widehat{\omega}\in\mathfrak{Q}^1-\mathfrak{Q}^2 \\ \mu_{\mathbb{K}^2(\widehat{\omega})}(\widehat{\pi}\widehat{\tau}) & ;\widehat{\omega}\in\mathfrak{Q}^2-\mathfrak{Q}^1 \\ \mu_{\mathbb{K}^1(\widehat{\omega})}(\widehat{\pi}\widehat{\tau})\wedge\mu_{\mathbb{K}^2(\widehat{\omega})}(\widehat{\pi}\widehat{\tau}) & ;\widehat{\omega}\in\mathfrak{Q}^1\cap\mathfrak{Q}^2 \end{cases}
\end{gathered} \tag{2}$$

Definition 22. For two popfhs-graphs, $\mathfrak{A}^1=(\mathfrak{A}^{1\star},\mathfrak{Q}^1,\mathbb{J}^1,\mathbb{K}^1)$ and $\mathfrak{A}^2=(\mathfrak{A}^{2\star},\mathfrak{Q}^2,\mathbb{J}^2,\mathbb{K}^2)$ with respect to $\mathfrak{A}^{1\star}=(\mathfrak{B}_1,\mathfrak{E}_2)$ and $\mathfrak{A}^{2\star}=(\mathfrak{B}_2,\mathfrak{E}_2)$. Let $\mathfrak{A}=\mathfrak{A}^1\uplus\mathfrak{A}^2$ be the join of $\mathfrak{A}^1$ and $\mathfrak{A}^2$ where $\mathfrak{A}=(\mathbb{J}^1\uplus\mathbb{J}^2,\mathbb{K}^1\uplus\mathbb{K}^2,\mathfrak{Q}^1\cup\mathfrak{Q}^2)$ is popfhs-set over $\mathfrak{B}=\mathfrak{B}_1\cup\mathfrak{B}_2$ and $\mathbb{K}=(\mathbb{K}^1\uplus\mathbb{K}^2,\mathfrak{Q}^1\cup\mathfrak{Q}^2)$ is popfhs-set over $\mathfrak{E}=\mathfrak{E}_1\cup\mathfrak{E}_2$ where

(1) $(\mathbb{J}^1\uplus\mathbb{J}^2,\mathfrak{Q}^1\cup\mathfrak{Q}^2)=(\mathbb{J}^1,\mathfrak{Q}^1)\cup(\mathbb{J}^2,\mathfrak{Q}^2)$

(2) $(\mathbb{K}^1\uplus\mathbb{K}^2,\mathfrak{Q}^1\cup\mathfrak{Q}^2)=$ $(\mathbb{K}^1,\mathfrak{Q}^1)\cup(\mathbb{K}^2,\mathfrak{Q}^2)$, if $\widehat{\pi}\widehat{\tau}\in\mathfrak{E}_1\cup\mathfrak{E}_2$

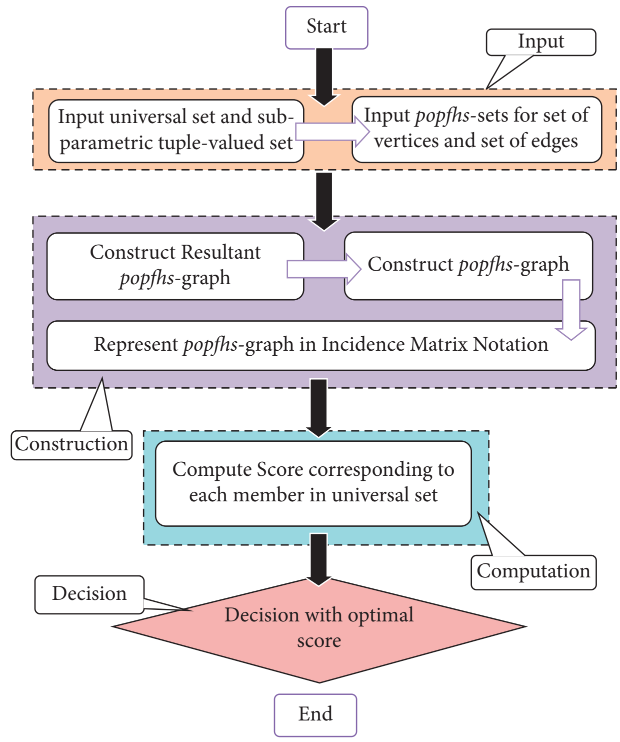


FIGURE 12: Step-wise brief graphical description of proposed algorithm.

Start
Input
(1) Input $\mathfrak{B}$ and $\mathfrak{Q}$ as universal set and set of parametric tuples, respectively.
(2) Input popfhs-sets $\{\mathbb{J}, \mathfrak{Q}\}$ and $\{\mathbb{K}, \mathfrak{Q}\}$.
Construction
(3) Construct popfhs-graph $\mathfrak{A} = (\mathfrak{A}^{\star}, \mathfrak{Q}, \mathbb{J}, \mathbb{K})$.
(4) Construct resultant popfhs-graph for $\widehat{\omega} = \wedge_{\kappa} \widehat{\omega}_{\kappa}\ \forall$ values of κ.
(5) Construct popfhs-graph $\mathbb{W}(\widehat{\omega})$ along with Ì-Matrix.
Computation
(6) Calculate score $\mathbb{S}_{\kappa}$ of $\mathfrak{C}_{\kappa}\ \forall$ values of κ and its average by the formula $\mathbb{S}_{\kappa} = \mathbb{T}_{\kappa} + \mathbb{I}_{\kappa} - \mathbb{F}_{\kappa} + \mu_{\kappa} + 1/4$.
Decision
(7) Opt $\mathfrak{C}_{\kappa}$ if $\mathfrak{C}_{\kappa} = \max_i \mathfrak{C}_i$.
(8) Opt any one of $\mathfrak{C}_{\kappa}$ if κ bears multiple values.
End

ALGORITHM 1: Optimal selection of candidate by using popfhs-graph.

when $\widehat{\omega} \in \mathfrak{Q}^1 \cap \mathfrak{Q}^2$ and $\widehat{\pi}\widehat{\tau} \in \mathfrak{E}$, and uncertain parts are as follows:

$$\begin{aligned}
\mathbb{T}_{\mathbb{K}^1 \uplus \mathbb{K}^2(\widehat{\omega})}(\widehat{\pi}\widehat{\tau}) &= \min\left\{\mathbb{T}_{\mathbb{J}^1(\widehat{\omega})}(\widehat{\pi}\widehat{\tau}), \mathbb{T}_{\mathbb{J}^2(\widehat{\omega})}(\widehat{\pi}\widehat{\tau})\right\},\\
\mathbb{I}_{\mathbb{K}^1 \uplus \mathbb{K}^2(\widehat{\omega})}(\widehat{\pi}\widehat{\tau}) &= \min\left\{\mathbb{I}_{\mathbb{J}^1(\widehat{\omega})}(\widehat{\pi}\widehat{\tau}), \mathbb{I}_{\mathbb{J}^2(\widehat{\omega})}(\widehat{\pi}\widehat{\tau})\right\},\\
\mathbb{F}_{\mathbb{K}^1 \uplus \mathbb{K}^2(\widehat{\omega})}(\widehat{\pi}\widehat{\tau}) &= \min\left\{\mathbb{F}_{\mathbb{J}^1(\widehat{\omega})}(\widehat{\pi}\widehat{\tau}), \mathbb{F}_{\mathbb{J}^2(\widehat{\omega})}(\widehat{\pi}\widehat{\tau})\right\},\\
\mu_{\mathbb{K}^1 \uplus \mathbb{K}^2(\widehat{\omega})}(\widehat{\pi}\widehat{\tau}) &= \min\left\{\mu_{\mathbb{J}^1(\widehat{\omega})}(\widehat{\pi}\widehat{\tau}), \mu_{\mathbb{J}^2(\widehat{\omega})}(\widehat{\pi}\widehat{\tau})\right\}.
\end{aligned} \tag{9}$$

Definition 23. The complement $\mathfrak{A}^c = (\mathfrak{A}^{\star\, c}, \mathfrak{Q}^c, \mathbb{J}^c, \mathbb{K}^c)$ of popfhs-graph $\mathfrak{A} = (\mathfrak{A}^{\star}, \mathfrak{Q}, \mathbb{J}, \mathbb{K})$ is a popfhs-graph for which $\widehat{\pi}, \widehat{\tau} \in \mathfrak{B}$ and $\widehat{\omega} \in \mathfrak{Q}$, and it satisfies the following conditions:

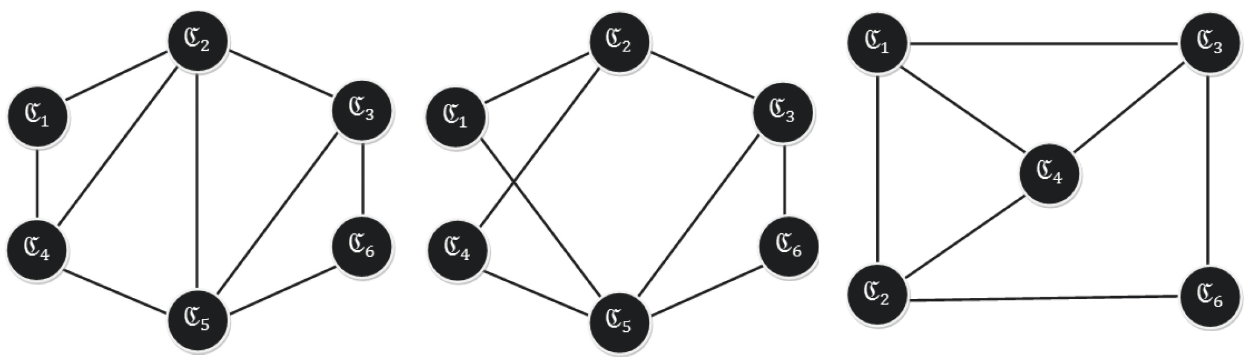
(1) $\mathfrak{Q}^c = \mathfrak{Q}$
(2) $\mathbb{J}^c(\widehat{\omega}) = \mathbb{J}(\widehat{\omega})$
(3) $\mathbb{T}_{\mathbb{K}^c(\widehat{\omega})}(\widehat{\pi}, \widehat{\tau}) = \mathbb{T}_{\mathbb{J}(\widehat{\omega})}(\widehat{\pi}) \wedge \mathbb{T}_{\mathbb{J}(\widehat{\omega})}(\widehat{\tau}) - \mathbb{T}_{\mathbb{K}(\widehat{\omega})}(\widehat{\pi}, \widehat{\tau})$
(4) $\mathbb{I}_{\mathbb{K}^c(\widehat{\omega})}(\widehat{\pi}, \widehat{\tau}) = \mathbb{I}_{\mathbb{J}(\widehat{\omega})}(\widehat{\pi}) \wedge \mathbb{I}_{\mathbb{J}(\widehat{\omega})}(\widehat{\tau}) - \mathbb{I}_{\mathbb{K}(\widehat{\omega})}(\widehat{\pi}, \widehat{\tau})$
(5) $\mathbb{F}_{\mathbb{K}^c(\widehat{\omega})}(\widehat{\pi}, \widehat{\tau}) = \mathbb{F}_{\mathbb{J}(\widehat{\omega})}(\widehat{\pi}) \wedge \mathbb{F}_{\mathbb{J}(\widehat{\omega})}(\widehat{\tau}) - \mathbb{F}_{\mathbb{K}(\widehat{\omega})}(\widehat{\pi}, \widehat{\tau})$
(6) $\mu_{\mathbb{K}^c(\widehat{\omega})}(\widehat{\pi}, \widehat{\tau}) = \mu_{\mathbb{J}(\widehat{\omega})}(\widehat{\pi}) \wedge \mu_{\mathbb{J}(\widehat{\omega})}(\widehat{\tau}) - \mu_{\mathbb{K}(\widehat{\omega})}(\widehat{\pi}, \widehat{\tau})$

Definition 24. If $\mathfrak{A}^c = \mathfrak{A}$ where $\mathfrak{A} = (\mathfrak{A}^{\star}, \mathfrak{Q}, \mathbb{J}, \mathbb{K})$ is a popfhs-graph, then $\mathfrak{A}$ is self-complementary.

Definition 25. If $\mathbb{K}(\widehat{\omega})$ is popfhs-graph of $\mathfrak{A}$, $\forall\, \widehat{\omega} \in \mathfrak{Q}$, then $\mathfrak{A}$ is complete, and one can write

Table 12: popfhs-graph $\mathfrak{A} = (\mathfrak{A}^{\star}, \mathfrak{Q}, \mathbb{J}, \mathbb{K})$.

$\mathbb{J}$	$\mathfrak{C}_1$	$\mathfrak{C}_2$	$\mathfrak{C}_3$	$\mathfrak{C}_4$	$\mathfrak{C}_5$	$\mathfrak{C}_6$	
$\widehat{\omega}_1$	(0.2, 0.1, 0.3, 0.4)	(0.3, 0.1, 0.1, 0.3)	(0.2, 0.1, 0.3, 0.5)	(0.1, 0.2, 0.1, 0.6)	(0.2, 0.1, 0.1, 0.5)	(0.2, 0.1, 0.2, 0.3)	
$\widehat{\omega}_2$	(0.1, 0.1, 0.3, 0.7)	(0.2, 0.3, 0.2, 0.7)	(0.1, 0.3, 0.2, 0.3)	(0.3, 0.1, 0.3, 0.8)	(0, 0, 0, 0)	(0.2, 0.3, 0.1, 0.7)	
$\widehat{\omega}_3$	(0.3, 0.4, 0.1, 0.7)	(0.2, 0.3, 0.1, 0.6)	(0.1, 0.1, 0.3, 0.5)	(0.2, 0.1, 0.2, 0.6)	(0.3, 0.2, 0.1, 0.7)	(0.2, 0.1, 0.3, 0.5)	
$\mathbb{K}$	$(\mathfrak{C}_1, \mathfrak{C}_2)$	$(\mathfrak{C}_1, \mathfrak{C}_3)$	$(\mathfrak{C}_1, \mathfrak{C}_4)$	$(\mathfrak{C}_1, \mathfrak{C}_5)$	$(\mathfrak{C}_2, \mathfrak{C}_3)$	$(\mathfrak{C}_2, \mathfrak{C}_4)$	$(\mathfrak{C}_2, \mathfrak{C}_5)$
$\widehat{\omega}_1$	(0.2, 0.2, 0.1, 0.2)	(0, 0, 1, 0)	(0.3, 0.2, 0.3, 0.3)	(0, 0, 1, 0)	(0.3, 0.1, 0.1, 0.3)	(0.2, 0.2, 0.2, 0.2)	(0.3, 0.1, 0.1, 0.3)
$\widehat{\omega}_2$	(0.2, 0.2, 0.3, 0.6)	(0.2, 0.1, 0.1, 0.2)	(0.2, 0.2, 0.1, 0.5)	(0, 0, 1, 0)	(0, 0, 1, 0)	(0.1, 0.2, 0.3, 0.6)	(0, 0, 1, 0)
$\widehat{\omega}_3$	(0.2, 0.1, 0.1, 0.5)	(0, 0, 1, 0)	(0, 0, 1, 0)	(0.1, 0.3, 0.2, 0.5)	(0.1, 0.2, 0.2, 0.4)	(0.2, 0.2, 0.1, 0.4)	(0, 0, 1, 0)
$\mathbb{K}$	$(\mathfrak{C}_2, \mathfrak{C}_6)$	$(\mathfrak{C}_3, \mathfrak{C}_4)$	$(\mathfrak{C}_3, \mathfrak{C}_5)$	$(\mathfrak{C}_3, \mathfrak{C}_6)$	$(\mathfrak{C}_4, \mathfrak{C}_5)$	$(\mathfrak{C}_5, \mathfrak{C}_6)$	
$\widehat{\omega}_1$	(0, 0, 1, 0)	(0, 0, 1, 0)	(0.2, 0.1, 0.2, 0.4)	(0.2, 0.2, 0.3, 0.2)	(0.1, 0.2, 0.3, 0.4)	(0.3, 0.1, 0.1, 0.3)	
$\widehat{\omega}_2$	(0.5, 0.1, 0.1, 0.5)	(0.2, 0.1, 0.4, 0.2)	(0, 0, 1, 0)	(0.3, 0.2, 0.1, 0.3)	(0, 0, 1, 0)	(0, 0, 1, 0)	
$\widehat{\omega}_3$	(0, 0, 1, 0)	(0, 0, 1, 0)	(0.2, 0.3, 0.1, 0.4)	(0.2, 0.3, 0.3, 0.4)	(0.2, 0.1, 0.2, 0.5)	(0.3, 0.1, 0.4, 0.3)	



(a) (b) (c)

Figure 13: Pictorial depiction of Table 1 with (a) $\mathbb{W}(\widehat{\omega}_1)$, (b) $\mathbb{W}(\widehat{\omega}_2)$, and (c) $\mathbb{W}(\widehat{\omega}_3)$.

Table 13: Score values with choice values.

	$\mathfrak{C}_1$	$\mathfrak{C}_2$	$\mathfrak{C}_3$	$\mathfrak{C}_4$	$\mathfrak{C}_5$	$\mathfrak{C}_6$	$\mathfrak{C}_\kappa$
$\mathfrak{C}_1$	0.250	0.300	0.225	0.175	0.200	0.250	1.400
$\mathfrak{C}_2$	0.300	0.250	0.200	0.325	0.225	0.225	1.525
$\mathfrak{C}_3$	0.225	0.200	0.250	0.150	0.200	0.325	1.400
$\mathfrak{C}_4$	0.175	0.325	0.150	0.250	0.175	0.250	1.325
$\mathfrak{C}_5$	0.200	0.225	0.200	0.175	0.250	0.150	1.200
$\mathfrak{C}_6$	0.250	0.225	0.325	0.250	0.150	0.250	1.450

Table 14: Advantageous aspects of proposed model over relevant existing models.

Models	Grade of true membership	Grade of false membership	Grade of neutrality	Mapping with single argument	Mapping with multiargument	Entitlement of possibility degree	Project selection ranking
fs-graph [30]	✓	×	×	✓	×	×	×
ifs-graph [31]	✓	✓	×	✓	×	×	×
pfs-graph [32]	✓	✓	✓	✓	×	×	×
$pofs$-set [33]	✓	×	×	✓	×	✓	×
$poifs$-set [34]	✓	✓	×	✓	×	✓	×
$poifs$-graph [35]	✓	✓	×	✓	×	✓	×
Proposed model	✓	✓	✓	✓	✓	✓	✓

$$\begin{aligned}\mathbb{T}_{\mathbb{K}(\widehat{\omega})}(\widehat{\pi}\widehat{\tau}) &= \min\{\mathbb{T}_{\mathbb{J}(\widehat{\omega})}(\widehat{\pi}), \mathbb{T}_{\mathbb{J}(\widehat{\omega})}(\widehat{\tau})\},\\ \mathbb{I}_{\mathbb{K}(\widehat{\omega})}(\widehat{\pi}\widehat{\tau}) &= \min\{\mathbb{I}_{\mathbb{J}(\widehat{\omega})}(\widehat{\pi}), \mathbb{I}_{\mathbb{J}(\widehat{\omega})}(\widehat{\tau})\},\\ \mathbb{I}_{\mathbb{K}(\widehat{\omega})}(\widehat{\pi}\widehat{\tau}) &= \min\{\mathbb{I}_{\mathbb{J}(\widehat{\omega})}(\widehat{\pi}), \mathbb{I}_{\mathbb{J}(\widehat{\omega})}(\widehat{\tau})\},\\ \mu_{\mathbb{K}(\widehat{\omega})}(\widehat{\pi}\widehat{\tau}) &= \min\{\mu_{\mathbb{J}(\widehat{\omega})}(\widehat{\pi}), \mu_{\mathbb{J}(\widehat{\omega})}(\widehat{\tau})\}.\end{aligned} \tag{10}$$

Definition 26. A popfhs-graph $\mathfrak{A} = (\mathfrak{A}^\star, \mathfrak{Q}, \mathbb{J}, \mathbb{K})$ is strong popfhs-graph if $\mathbb{K}(\widehat{\omega})$ is popfhs-graph of $\mathfrak{A}$ and $\forall\, \widehat{\omega} \in \mathfrak{Q}$.

Theorem 8. *For two strong popfhs-graphs, $\mathfrak{A}^1 = (\mathfrak{A}^{1\star}, \mathfrak{Q}^1, \mathbb{J}^1, \mathbb{K}^1)$ and $\mathfrak{A}^2 = (\mathfrak{A}^{2\star}, \mathfrak{Q}^2, \mathbb{J}^2, \mathbb{K}^2)$ with respect to $\mathfrak{A}^{1\star} = (\mathfrak{B}_1, \mathfrak{E}_2)$ and $\mathfrak{A}^{2\star} = (\mathfrak{B}_2, \mathfrak{E}_2)$; then $\mathfrak{A}^1[\mathfrak{A}^2]$; the composition of $\mathfrak{A}^1$ and $\mathfrak{A}^2$ is strong popfhs-graph.*

Theorem 9. *For two strong popfhs-graphs, $\mathfrak{A}^1 = (\mathfrak{A}^{1\star}, \mathfrak{Q}^1, \mathbb{J}^1, \mathbb{K}^1)$ and $\mathfrak{A}^2 = (\mathfrak{A}^{2\star}, \mathfrak{Q}^2, \mathbb{J}^2, \mathbb{K}^2)$ with respect to $\mathfrak{A}^{1\star} = (\mathfrak{B}_1, \mathfrak{E}_2)$ and $\mathfrak{A}^{2\star} = (\mathfrak{B}_2, \mathfrak{E}_2)$; then $\mathfrak{A}^1 \times_{\mathbb{P}} \mathfrak{A}^2$; the Cartesian product of $\mathfrak{A}^1$ and $\mathfrak{A}^2$ is strong popfhs-graph.*

Definition 27. The complement $\mathfrak{A}^c = (\mathfrak{A}^{\star\, c}, \mathfrak{Q}^c, \mathbb{J}^c, \mathbb{K}^c)$ of strong popfhs-graph $\mathfrak{A} = (\mathfrak{A}^\star, \mathfrak{Q}, \mathbb{J}, \mathbb{K})\ \forall\, \widehat{\omega} \in \mathbb{J}, \widehat{\pi}, \widehat{\tau} \in \mathfrak{B}$ is given by

(1) $\mathfrak{Q}^c = \mathfrak{Q}$

(2) $\mathbb{J}^c(\widehat{\omega})(\widehat{\pi}) = \mathbb{J}(\widehat{\omega})(\widehat{\pi})$

(3) $\mathbb{T}_{\mathbb{K}^c(\widehat{\omega})}(\widehat{\pi}, \widehat{\tau}) = \begin{cases} \min\{\mathbb{T}_{\mathbb{J}(\widehat{\omega})}(\widehat{\pi}), \mathbb{T}_{\mathbb{J}(\widehat{\omega})}(\widehat{\tau})\} & ; \mathbb{T}_{\mathbb{K}(\widehat{\omega})}(\widehat{\pi}, \widehat{\tau}) = 0 \\ 0 & ; \mathbb{T}_{\mathbb{K}(\widehat{\omega})}(\widehat{\pi}, \widehat{\tau}) > 0 \end{cases}$

(4) $\mathbb{I}_{\mathbb{K}^c(\widehat{\omega})}(\widehat{\pi}, \widehat{\tau}) = \begin{cases} \min\{\mathbb{I}_{\mathbb{J}(\widehat{\omega})}(\widehat{\pi}), \mathbb{I}_{\mathbb{J}(\widehat{\omega})}(\widehat{\tau})\} & ; \mathbb{I}_{\mathbb{K}(\widehat{\omega})}(\widehat{\pi}, \widehat{\tau}) = 0 \\ 0 & ; \mathbb{I}_{\mathbb{K}(\widehat{\omega})}(\widehat{\pi}, \widehat{\tau}) > 0 \end{cases}$

(5) $\mathbb{F}_{\mathbb{K}^c(\widehat{\omega})}(\widehat{\pi}, \widehat{\tau}) = \begin{cases} \min\{\mathbb{F}_{\mathbb{J}(\widehat{\omega})}(\widehat{\pi}), \mathbb{F}_{\mathbb{J}(\widehat{\omega})}(\widehat{\tau})\} & ; \mathbb{F}_{\mathbb{K}(\widehat{\omega})}(\widehat{\pi}, \widehat{\tau}) = 0 \\ 0 & ; \mathbb{F}_{\mathbb{K}(\widehat{\omega})}(\widehat{\pi}, \widehat{\tau}) > 0 \end{cases}$

(6) $\mu_{\mathbb{K}^c(\widehat{\omega})}(\widehat{\pi}, \widehat{\tau}) = \begin{cases} \min\{\mu_{\mathbb{J}(\widehat{\omega})}(\widehat{\pi}), \mu_{\mathbb{J}(\widehat{\omega})}(\widehat{\tau})\} & ; \mu_{\mathbb{K}(\widehat{\omega})}(\widehat{\pi}, \widehat{\tau}) = 0 \\ 0 & ; \mu_{\mathbb{K}(\widehat{\omega})}(\widehat{\pi}, \widehat{\tau}) > 0 \end{cases}$

Theorem 10. *The complement $\mathfrak{A}^c = (\mathfrak{A}^{\star\, c}, \mathfrak{Q}^c, \mathbb{J}^c, \mathbb{K}^c)$ of strong popfhs-graph $\mathfrak{A} = (\mathfrak{A}^\star, \mathfrak{Q}, \mathbb{J}, \mathbb{K})\ \forall\, \widehat{\omega} \in \mathbb{J}, \widehat{\pi}, \widehat{\tau} \in \mathfrak{B}$ is strong popfhs-graph.*

Theorem 11. *If $\mathfrak{A} = (\mathfrak{A}^\star, \mathfrak{Q}, \mathbb{J}, \mathbb{K})$ and its complement $\mathfrak{A}^c = (\mathfrak{A}^{\star\, c}, \mathfrak{Q}^c, \mathbb{J}^c, \mathbb{K}^c)$ are strong popfhs-graphs $\forall\, \widehat{\omega} \in \mathbb{J}, \widehat{\pi}, \widehat{\tau} \in \mathfrak{B}$, then $\mathfrak{A} \cup \mathfrak{A}^c$ is complete popfhs-graph.*

5. Application in Decision-Making

In order to validate the proposed study, an algorithm-based application is discussed for reliable decision-making process (The brief graphical description of steps involved in Algorithm 1 is presented in Figure 12).

Example 6. Suppose an organization intends to recruit a candidate to fill a vacant post of assistant manager. Six candidates, that is, $\mathfrak{B} = \{\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3, \mathfrak{C}_4, \mathfrak{C}_5, \mathfrak{C}_6\}$, have been scrutinized by recruitment committee. The committee further requires evaluation to select one of these candidates. The evaluation indicators are qualification (β_1), relevant experience (β_2), and computer skill (β_3). Their subparametric disjoint sets are $\mathfrak{Q}_1 = \{\beta_{11}\}$, $\mathfrak{Q}_2 = \{\beta_{21}, \beta_{22}, \beta_{23}\}$, and $\mathfrak{Q}_3 = \{\beta_{31}\}$, respectively, such that $\mathfrak{Q} = \mathfrak{Q}_1 \times \mathfrak{Q}_2 \times \mathfrak{Q}_3 = \{\widehat{\omega}_1, \widehat{\omega}_2, \widehat{\omega}_3\}$ and $\mathfrak{A} = \{(\mathbb{W}, \mathfrak{Q})\} = \{(\mathbb{W}(\widehat{\omega}_1)), (\mathbb{W}(\widehat{\omega}_2)), (\mathbb{W}(\widehat{\omega}_3))\}$ are popfhs-graph. This selection is accomlished by proposing an algorithm (i.e. Algorithm 1 which is presented in Figure 1)

The graphical explanation of Table 12 is provided in Figure 13.

The popfhs-graphs $\mathbb{W}(\widehat{\omega}_1), \mathbb{W}(\widehat{\omega}_2)$, and $\mathbb{W}(\widehat{\omega}_3)$ with respect to subparametric values are given in Table 12 and stated in Figure 2. The I-Matrices of popfhs-graphs are

$$\mathbb{W}(\widehat{\omega}_1) = \begin{pmatrix} (0,0,0,0) & (0.2,0.2,0.1,0.2) & (0,0,0,0) & (0.3,0.2,0.3,0.3) & (0,0,0,0) & (0,0,0,0) \\ (0.2,0.2,0.1,0.2) & (0,0,0,0) & (0.3,0.1,0.1,0.3) & (0.2,0.2,0.2,0.2) & (0.3,0.1,0.1,0.3) & (0,0,0,0) \\ (0,0,0,0) & (0.3,0.1,0.1,0.3) & (0,0,0,0) & (0,0,0,0) & (0.2,0.1,0.2,0.4) & (0.2,0.2,0.3,0.2) \\ (0.3,0.2,0.3,0.3) & (0.2,0.2,0.2,0.2) & (0,0,0,0) & (0,0,0,0) & (0.1,0.2,0.3,0.4) & (0,0,0,0) \\ (0,0,0,0) & (0.3,0.1,0.1,0.3) & (0.2,0.1,0.2,0.4) & (0.1,0.2,0.3,0.4) & (0,0,0,0) & (0.3,0.1,0.1,0.3) \\ (0,0,0,0) & (0,0,0,0) & (0.2,0.2,0.3,0.2) & (0,0,0,0) & (0.3,0.1,0.1,0.3) & (0,0,0,0) \end{pmatrix},$$

$$\mathbb{W}(\widehat{\omega}_2) = \begin{pmatrix} (0,0,0,0) & (0.2,0.2,0.3,0.6) & (0.2,0.1,0.1,0.2) & (0.2,0.2,0.1,0.5) & (0,0,0,0) & (0,0,0,0) \\ (0.2,0.2,0.3,0.6) & (0,0,0,0) & (0,0,0,0) & (0.1,0.2,0.3,0.6) & (0,0,0,0) & (0.5,0.1,0.1,0.5)) \\ (0.2,0.1,0.1,0.2) & (0,0,0,0) & (0,0,0,0) & (0.2,0.1,0.4,0.2) & (0,0,0,0) & (0.3,0.2,0.1,0.3)) \\ (0.2,0.2,0.1,0.5) & (0.1,0.2,0.3,0.6) & (0.2,0.1,0.4,0.2) & (0,0,0,0) & (0,0,0,0) & (0,0,0,0) \\ (0,0,0,0) & (0,0,0,0) & (0,0,0,0) & (0,0,0,0) & (0,0,0,0) & (0,0,0,0) \\ (0,0,0,0) & (0.5,0.1,0.1,0.5) & (0.3,0.2,0.1,0.3) & (0,0,0,0) & (0,0,0,0) & (0,0,0,0) \end{pmatrix},$$

$$\mathbb{W}(\widehat{\omega}_3) = \begin{pmatrix} (0,0,0,0) & (0.2,0.1,0.1,0.5) & (0,0,0,0) & (0,0,0,0) & (0.1,0.3,0.2,0.5) & (0,0,0,0) \\ (0.2,0.1,0.1,0.5) & (0,0,0,0) & (0.1,0.2,0.2,0.4) & (0.2,0.2,0.1,0.4) & (0,0,0,0) & (0,0,0,0) \\ (0,0,0,0) & (0.1,0.2,0.2,0.4) & (0,0,0,0) & (0,0,0,0) & (0.2,0.3,0.1,0.4) & (0.2,0.3,0.3,0.4) \\ (0,0,0,0) & (0.2,0.2,0.1,0.4) & (0,0,0,0) & (0,0,0,0) & (0.2,0.1,0.2,0.5) & (0,0,0,0) \\ (0.1,0.3,0.2,0.5) & (0,0,0,0) & (0.2,0.3,0.1,0.4) & (0.2,0.1,0.2,0.5) & (0,0,0,0) & (0.3,0.1,0.4,0.3) \\ (0,0,0,0) & (0,0,0,0) & (0.2,0.3,0.3,0.4) & (0,0,0,0) & (0.3,0.1,0.4,0.3) & (0,0,0,0) \end{pmatrix}. \quad (11)$$

The incidence matrix for resultant $popfhs$-graph can be calculated by $\widehat{\omega} = \widehat{\omega}_1 \wedge \widehat{\omega}_2 \wedge \widehat{\omega}_3$, which is given by

$$\mathbb{W}(\widehat{\omega}) = \begin{pmatrix} (0,0,0,0) & (0.2,0.1,0.3,0.2) & (0,0,0.1,0) & (0,0,0.3,0) & (0,0,0.2,0) & (0,0,0,0) \\ (0.2,0.1,0.3,0.2) & (0,0,0,0) & (0,0,0.2,0) & (0.1,0.2,0.2,0.2) & (0,0,0.1,0) & (0,0,0.1,0) \\ (0,0,0.1,0) & (0,0,0.2,2) & (0,0,0,0) & (0,0,0.4,0) & (0,0,0.2,0) & (0.2,0.2,0.3,0.2) \\ (0,0,0.3,0) & (0.1,0.2,0.2,0.2) & (0,0,0.4,0) & (0,0,0,0) & (0,0,0.3,0) & (0,0,0,0) \\ (0,0,0.2,0) & (0,0,0.1,0) & (0,0,0.2,0) & (0,0,0.3,0) & (0,0,0,0) & (0,0,0.4,0) \\ (0,0,0,0) & (0,0,0.1,0) & (0.2,0.2,0.3,0.2) & (0,0,0,0) & (0,0,0.4,0) & (0,0,0,0) \end{pmatrix}. \quad (12)$$

Score and its average values are given in Table 13.

Since according to Table 13, the maximum average score value for $\mathfrak{C}_2$ is 1.525, so the candidate $\mathfrak{C}_2$ is selected.

6. Comparison Analysis

The problem of project selection for investment selection is of great importance for investors. A very few literature exists in this regard under uncertain environments (i.e., fuzzy set-like environments). But there exists no literature regarding this project under soft set-like and possibility soft set-like environments. It is an advantageous aspect of the proposed study over the existing relevant models that an important problem has been discussed through this study. Due to lack of relevant literature, the numerical comparison is not possible, but we present the comparison of the proposed model with some existing relevant models on structural basis to depict its advantageous aspects and flexibility. Table 14 presents this structural comparison.

7. Conclusions

In this research, authors have managed the real-world decision-making situation that demands: (i) the categorization of opted parameters into their respective disjoint subclasses having their relevant attributive values, (ii) the consideration of multiargument parameterization in the domain of approximate mapping to have reliable approximation of alternatives, and (iii) a mode for the assessment of uncertain nature of approximate elements to have level of acceptance, collectively with the development of novel context of popfhs-graph. As a conceptual framework, essential rudiments, operations, and products are characterized with the help of elaborated instances. A real-world decision-making problem from human resource management for the optimal selection of candidate is resolved with the proposal of an intelligent algorithm. Since the sum of values of uncertain components in proposed model is considered within [0,1], but it is not sufficient for the situations where decision-makers provide their opinions as uncertain values whose sum exceeds 1; therefore, future work may include the extension of this work to possibility neutrosophic hypersoft setting to deal with above describe limitation. Moreover, many other notions of classical graph may also be characterized by utilizing the proposed context.

Data Availability

This study has no associated data.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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